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Changing Teaching Mathematics in the Changing Society

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Contents

Editorial

This monothematic issue focuses on mathematics and its teaching both from the point of view of pupils and (student) teachers. The title *Changing Teaching Mathematics in the Changing Society* reflects our belief that teaching mathematics must reflect the changes which occur in the society. The scope of mathematics education research is very wide. Rather than focusing on one particular issue we decided to provide the readers with glimpses at different corners of mathematics teaching research. Thus, the issue contains four theoretical papers focusing on theoretical framework underpinning research, on the goals of teaching mathematics and on the possible means to reach these goals. Next, four empirical papers are included, which focus on pupils, teachers and student teachers.

The issue is opened by two studies complementing to a certain extent each other. In the opening paper *Societal Mediation of Mathematical Cognition and Learning* Wollf-Michael Roth provides an outline of cultural-historical activity theory which could be used as an inclusive theoretical framework allowing us to understand the totality of levels that characterizes the participation in mathematical practices or in practices that use mathematics as a means of production.

On the other hand, Miroslav Rendl and Stanislav Štech in their paper *Should Learning* (*Mathematics*) *at School Aim at Knowledge or at Competences?* look critically at so called situated learning and the use of real life problems as THE answer to ineffective teaching. They also call into question the existence of transfer of knowledge gained in school mathematics into everyday life.

The overarching goal of research in mathematics education is the improvement of the teaching of mathematics so that it leads to good knowledge of mathematics by the majority of pupils and to their ability to use mathematics not only in their academic careers but also in their everyday life. Throughout the years, the objects of study were individuals and their learning, groups of pupils in their classrooms and also (student) teachers. Milan Hejný's paper *Cognitive Goals of the Teaching of Mathematics and Scheme-oriented Education* is based on views presented by H. Freudenthal, E. Fischbein and many others which postulate that the main way to improve the teaching of mathematics is the change of a teacher's educational style towards the development of pupils' creativity and intellectual autonomy. In M. Hejný's interpretation, it means towards scheme-oriented education based on the theory of generic models. Both concepts are explained in the paper and serve as a theoretical background for the development of a diagnostic tool which can be used for the characterisation of a mathematics teacher's educational style described and illustrated by Darina Jirotková in her paper called *Tool for Diagnosing the Teacher's Educational Style in Mathematics*.

In their joint paper *Contribution of Geometry to the Goals of Education in Mathematics* Milan Hejný and Darina Jirotková show how through visualisation, geometry can mediate understanding of some demanding arithmetic and algebraic concepts, relationships, processes and situations for pupils. This thesis is explained by the method of genetic parallel and of a didactic analysis of two educationally interesting problem situations. Theoretical considerations are illustrated by several real experiences.

Next, our attention will shift towards student teachers and their professional knowledge. Naďa Vondrová and Jana Žalská explore the question *Do Student Teachers Attend to Mathematics Specific Phenomena when Observing Mathematics Teaching on Video?* The empirical study of thirty student teachers brings insight not only in the participants' ability to notice but also in categories of content-related observable aspects of teaching. The results have implications for both teacher development design and effective teaching practice.

The following two papers focus on pupils. Andrea Gellert and Heinz Steinbring in their empirical research *Dispute in Mathematical Classroom Discourse – "No go" or Chance for Fundamental Learning?* look into the discourse based on disputed points which turned out to be a promising way of supporting mathematics oriented discussions, a deeper understanding of ambiguous learning situations and the chances for fundamental learning. Illustrations are given.

The concluding paper *Mathematical Perception of Pupils and Teachers* considers mathematics from the point of view of pupils and their teachers. The authors, Isabella Pavelková and Vladimír Hrabal, show the possibilities of increasing the teacher's professional competencies using focused self-reflection on the basis of confrontation of his or her own ideas, expectations and observations with data obtained mostly from the statements of pupils. The paper is quantitative in nature.

The issue is complemented by two contributions which present what is 'going on' in the field of mathematics education. Jarmila Novotná and Alena Hošpesová describe the project *The Learner's Perspective Study* and the resulting books published by Sense Publishers. They aim at international comparative research in mathematics education, namely at documenting, comparing and contrasting teaching practices in several countries. Finally, Jarmila Novotná describes international *Symposium on Elementary Maths Teaching* (*SEMT*) which is a biannual conference for researchers and teachers interested in elementary mathematics teaching organised since 1991 in the Czech Republic.

I hope that this special issue will bring some insight and inspiration not only to people in mathematics education but also to readers from other scientific fields interested in teaching and learning.

Naďa Vondrová

Societal Mediation of Mathematical Cognition and Learning[1]

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Abstract: Cultural-historical activity theory, originally developed by A. N. Leontjew in the 1960s and 70s, has been experiencing a revival in Western scholarship over the past two decades. Whereas the analytical category "activity" was offered up as a category that denotes systems that contribute to the production of things to meet generalized, societal needs, it tends to be used in mathematics education to refer to simple tasks and exercises – e.g., calculating the areas of geometrical figures. In this paper I argue, drawing on empirical examples from my own research, that the very strengths of cultural-historical activity theory are not realized, which lie in the affordance to integrate macro-sociological with micro-psychological dimensions of cognition and learning. Moreover, in the current way the theory is used, its potential as a critical theory is given short shrift in the service of emancipatory efforts that uproot the tendency for schooling to reproduce bourgeois society and its class structure.

Keywords: activity theory, mediation, reproduction, unit analysis

1 Introduction

The research on knowing and learning in mathematics education, consistent with the dominant constructivist epistemology, tends to focus on individuals and school classes while they do tasks and exercises – though there are indeed other efforts, such as ethnomathematics (D'Ambrosio, 2008) and critical mathematics education (e.g., Skovsmose, 2011), that are concerned with societal mediations of mathematical cognition and learning. The purpose of this article is to sketch out cultural-historical activity theory in support of an argument for using an inclusive theoretical framework that allows us to understand the totality of levels that characterizes the participation in mathematical practices or in practices that use mathematics as a means of production. Consider the following two vignettes, on which I draw as empirical examples in the course of articulating the theory. As part of a two-year ethnographic effort, my research team videotaped seventh-grade students in the course of their regular mathematics classes and while participating in an experimental science curriculum that made it possible for students to participate in an already existing environmental activist movement.

[1] The empirical data presented in this study were prepared with support from various grants from the Social Sciences and Humanities Research Council of Canada and the Natural Sciences and Engineering Research Council of Canada.

Vignette 1: In one of the mathematics classes, the teacher had decided to support the teaching of science by introducing the students to graphs and graphing. The teacher had prepared a task sheet containing several columns with numbers. He explained that the task was to find relationships between any two columns of numbers of the students' choice. He also prompted the students by telling them that they "had learned about a variety of graphing techniques," and then he listed, among others, pie charts, bar graphs, and scatter plots. After handing out the sheets, he reminded his students to use pencils, which would allow them to make corrections to their work if needed. The camera follows Jamie, "one of the better students," and Davie. Jamie settles down to begin working on the task, whereas Davie queries the teacher as soon as the latter passes their desk.

Davie: What are we supposed to do? Like when is, um, um, like a bar graph? (*Points to an example of a bar graph in the book in front of him.*)
Teacher: A scatter plot graph and choose the speed and one of these other categories.
Davie: We are comparing this (*he points to one column on the task sheet*) and this (*points to a second column*) and this (*points to a third column*).
Teacher: So make a scatter plot that compares two things.
Davie: But how do I make a scatter plot? (*He restlessly gets out of his seat and appears to move away from it.*)
Teacher: You are not going to do it? (*He gently pushes Davie back into his seat.*)
Davie: Jamie is going to do it.
Teacher: And you are going to do the rest of it? Okay!

"What do we have to do?," Jamie asks Davie. "I thought you knew what to do," the latter responds. Jamie orients to the task by beginning to draw axes on his lined paper, whereas Davie gets up and throws some crumbled paper at another group of students. He then takes his bottle of lemonade and shares it with other students in the class. Every now and then he returns, watches Jamie for a while, and then continues walking about the classroom. At one instant while Davie observes Jamie, the teacher passes by their desk. Jamie uses it as a way of addressing the teacher: "We don't get this." But the teacher moves on. When he returns, Davie, who has been talking to a neighbor, again watches Jamie. The teacher asks Davie, "Are you contributing?" to which the latter responds: "Some." The teacher moves on again when a student from another group asks, "What kind of graph are you using?" Davie answers, "I don't know how to do it." He subsequently continues walking about the classroom, talking to others, sharing his drink, and at times watching them at work on the task. In the end, Davie will have spent contributing less than 2 of the 26 minutes that the teacher had allotted to the graphing task. The teacher and the teaching intern later tell me that Davie has "attention deficit hyperactive disorder" and "severe writing problems" and has been identified, for these reasons, to be "learning disabled."

In this instance, we observe a student (Davie) who, by all accounts, behaves in the mathematics class consistent with the label that the school and psychologists have pinned on him. Davie engages little with the task and, as a result, does not produce what the teacher wants him to produce. Repeatedly in the course of this lesson, Davie says that he does not know what to do and that his partner Jamie is going to do it. He spends much of the lesson "off-task," talking to other students, sharing his drink, and sometimes interrupting others by bugging them. He apparently knows little and, during this lesson, does not augment what he knows by learning what to do and how to do it when requested to analyze tables of numbers.

Vignette 2: The same class of student participated in an experimental curriculum that I had designed and taught, in this instance, together with the teaching intern. The curriculum allowed students to participate in and realize environmental activism. Encouraged by the call that an environmentalist activist group published in the local newspaper – concerning help required in finding out and doing something about the sorry state of the creek that drains the watershed in which the municipality is located – Davie and his peers decided to produce knowledge about the creek and to publish what they learn at an open-house event, which the environmentalists planned later that year. Davie and Stevie, another "learning disabled" student, emerged from this unit as widely recognized experts. That is, this science unit, which essentially existed in creating knowledge and services that were contributed to this farming community by working on a community-relevant problem, set up situations in which Davie and Stevie turn out to be functional and literate individuals. Nobody watching the videotapes without knowing their background actually would label them to be learning disabled. More pertinently, so-called learning-disabled students developed tremendous forms of knowledgeability with respect to scientific and mathematical representations that depicted Hagan Creek in its current state at the time. In this unit, they became experts on environmentalism in their school.

Davie and Stevie not only became competent participants in environmentalism but also were among the first to present their findings in another class of the same grade. They taught the teacher of that class on how to design and conduct research. They also signed up as mentors of their peers from another class when these, too, came to participate in environmentalism. Here, for example, although Davie and Stevie had not done well on the task of producing graphs during their mathematics lessons, they in fact tutored their peers on how to transform data that they had collected (Figure 1a) into a graph that showed the relationship between the width of the creek and the speed of the water at two different locations (Figure 1b).

My research team collected a lot of evidence for the mathematical competencies that were produced while Davie and Stevie assisted their peers (Figure 1b). That such would be the case is not self-evident, for even professors teaching undergraduate courses produce erroneous translations from one form of representation to a graph and vice versa (Roth & Bowen, 1999). However, the present results are consistent with other research: when students learn mathematical representations in the context of school-based tasks of their own design, which allow them certain levels of control over the object and the means to produce knowledge claims that they have to "sell" to their peers and teachers, then they become highly competent (e.g., Roth & Bowen, 1994).

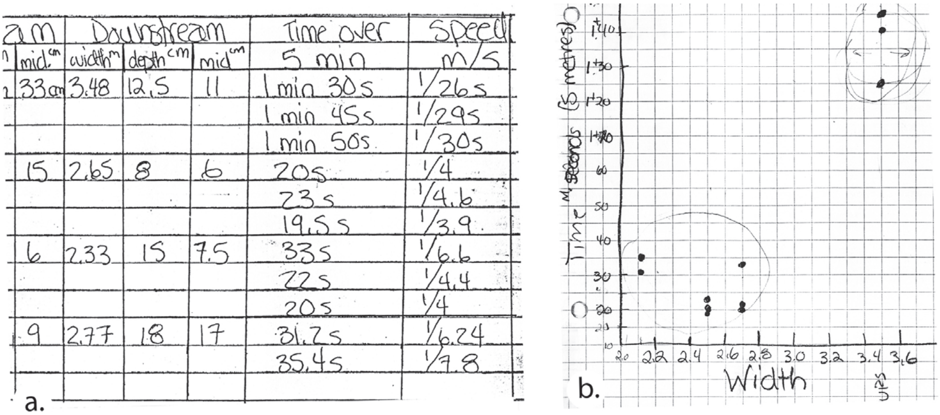

Figure 1 Davie has tutored a group of peers in another seventh-grade class in producing this graph from the data they collected at Hagan Creek.

Our study ended when Davie and Stevie took part in the open-house event presenting, alongside the environmental activists, their own mathematical representations that articulate features of the creek. Just as the water technician working at a local farm and member of the environmental activist group represents a graph exhibiting water levels and rainfall in the course of a year, the two students present their results in the form of graphs. Our video recordings show that adults actively engaged Davie while he talked about the findings of his and Stevie's research presented in the mathematical representations. It was precisely for this instant that they had worked: to contribute, in response to the environmentalists' call for community participation, to their municipality by informing its members of the not-well-known sorry state of the creek.

The differences between the two courses are striking. In one, Davie, Stevie, and students like them exhibit behaviors and performances of the type that warranted the school to treat them as "learning disabled." The result of their participation in the tasks led the school system to regard them as deficient. Whereas this also meant that the students received "special help," this help required them to leave their classroom context, and, therefore, interrupted any relation that they have had with others and which normally supported them in their participation. On the other hand, our videotapes testify to the tremendous forms of knowledgeability exhibited and developed wherever and whenever the two students appear. They were teaching and otherwise assisting others in doing research and representing research by means of mathematical representations. How are such striking differences possible given that the two vignettes derive from the same time period? Going theories have difficulties explaining these differences, because these pin knowledge and knowing to the individual mind. Cultural-historical activity theory, on the other hand, is capable of explaining the differences, which arise, in part, from the very structures of society. In the following, I sketch this theory by exemplifying pertinent aspects with materials from the same study. The theory does not only explain the differences but also provides the tools for critiquing and changing the (societal) conditions in which students learn.

2 Cultural-Historical Activity Theory

Cultural-historical activity theory was created to understand the specific nature of human consciousness, the psychic reflection of material reality that emerges in the course of labor and produces ordinary and everyday society in the way we are familiar with it (Leontyev, 1981). It theorizes the psyche as having developed in the course of evolution, where existing animal forms of cooperative behavior led to a generalized division of labor during the time of anthropogenesis. Generalized division of labor means to participate in labor activity in a way that a split is achieved between the goals of immediate actions and the motives of the overall activity. Thus, in the early forms of division of labor, individuals contributed to the provision of food in various ways: in the collective hunt, some participated as beaters and hunters made the kills; others remained at the campsite to maintain the fire or produce the tools required in the hunt. There is therefore a network of activities that together meet the dietary needs of the group (Figure 2). By contributing to the generalized control over collective conditions, hominids expanded their individual control over life conditions and provision of personal needs. This, then, created an important double relation: humans create the conditions under which they live. This also means that they can change these conditions. In the course of cultural history of human collectivities, the increasing division of labor led to societies that exist in the form of highly diversified networks of activities.

2.1 Definition of activity and activity levels

To begin, a clarification is in order. In English, as in other languages, two different German (Russian) concepts important to activity theory, and which must not be conflated (Leontjew, 1982), are rendered by the same term *activity*. Thus, *Tätigkeit* (*deyatel'nost'*) refers to a collective formation that produces something important to the generalized provision of human needs at the level of society. Typical activities include farming, producing tools, and environmental activism. An *Aktivität* (*aktivnost'*), on the other hand, refers to a vital process – e.g., in narrower form the functions of the brain or the vocal cords. In English, both of these terms are referred to as "activity." From here on, the term activity is reserved for use according to the former sense. Activities have conscious societal object/motives. Applied to the educational context, activity in the sense of *Tätigkeit* (*deyatel'nost'*) refers to schooling whereas activity in the sense of *Aktivität* (*aktivnost'*) refers to enacting tasks that students do not necessarily understand (Roth & Lee, 2007). In schooling, the practical object/motive are the diplomas that permit/hinder students to enter

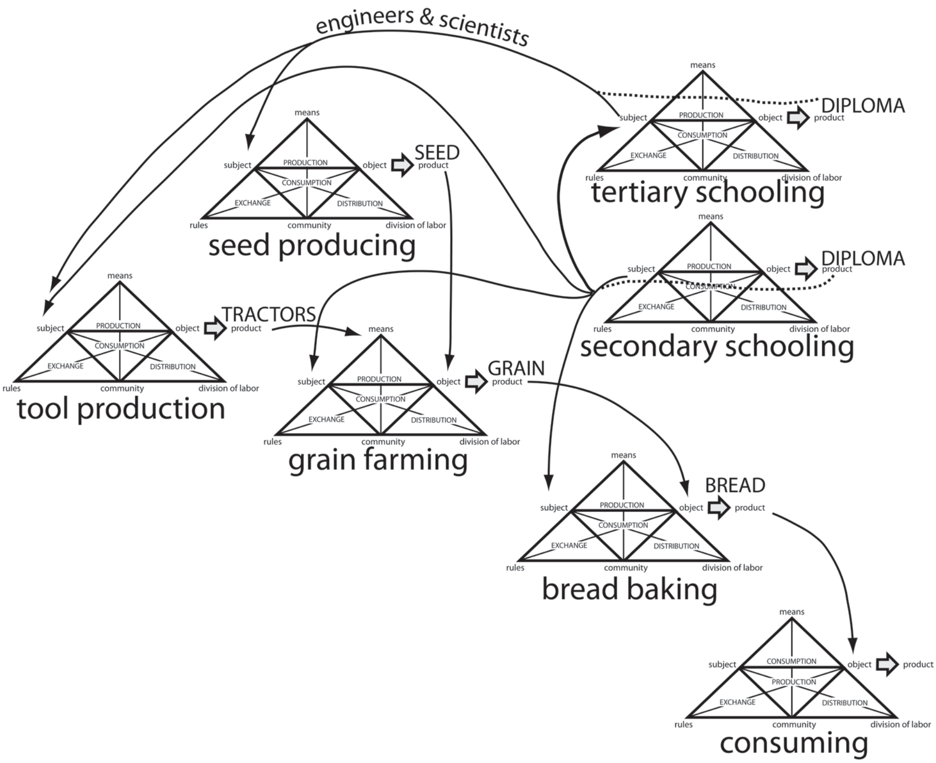

Figure 2 Society is network of activity systems, each of which contributes to meeting the satisfaction generalized human needs (here food). Division of labor can lead to the generation of new forms of activity from existing activities; and new forms of need may be created, also leading to new forms of activity. This synchronic perspective emphasizes the structural dimensions.

other forms of activity (see below). In the theory, *activity* is an analytic category that circumscribes the smallest unit that makes sense. Although we may identify *moments* of an activity (i.e., apparent sub-structures) these *moments* do not make sense independently of the total activity, which also manifests itself in other moments. That is, we must not think the system in terms of the sum of its manifestations because the relation arising from a summation would produce only *external*, arbitrary relations between the different manifestations. Cultural-historical activity theory was designed to think the phenomenon in terms of the *inner* relations of activity.

It takes a number of goal-directed *actions* to realize an activity. That is, an activity exists only because there are actions that realize it, but the action is mobilized only because there is an activity toward which it is directed. Schooling requires students to engage in tasks (e.g., constructing a graph given a table of numbers), but these tasks are chosen precisely because they realize schooling (i.e., lead to grades, grade reports, and diplomas). The activity and the actions that realize it, therefore, are mutually constitutive: they stand in a whole–part relation. An action (e.g., producing a graph) may contribute to different activities (e.g., a task in mathematics class vs. an environmentalist exhibit); it therefore takes its specific sense from the activity. This allows us to understand that Davie knows and understands what to do to produce line graphs and bar charts as part of the environmentalist activity, whereas he "does not know how to do it" in mathematics class. The action of making a coordinate system would have been different if Davie, Jamie, and Stevie had had access to one of the computers available in the mathematics class rather than to paper-and-pencil only.

Actions are realized, in turn, by contextually determined operations. Jamie may have had the goal to draw the axes of the graph but he did not have to think about how to hold the pen, which constitutes an operation. But this operation is mobilized only because there is an action that calls for it. An action and the operations that realize it are mutually constitutive. In the course of individual development, an action – e.g., scaling and labeling the axes of a graph – may become an operation, no longer requiring conscious attention. In fact, tools may embody actions in a crystallized way, such as when a computer makes axes automatically once students choose "line graph" from the appropriate menu.

The advantage of cultural-historical activity theory over other theories is this: it recognizes that the analysis of any instant of object-oriented human praxis requires the conjoint attention to all three levels simultaneously because of the mutually constitutive relations that bind them together. We may not analyze, therefore, what students do when tasked to produce a graph from data presented in tabular form independently of the activity of schooling, on the one hand, and independently of the range of the embodied operations that students have developed as part of their personal histories, on the other hand. Moreover, two instances of praxis separated in time cannot be compared without taking into account the historical nature of societal activity, so that doing graphs or mathematical modeling phenomena in the 1980s, in the absence of rapid and powerful personal computers, was very different from what it is or can be today: a simple click on a pull-down menu and selection of the option "line graph" is all that is required to plot the data presented in two columns of a spreadsheet.

2.2 The structure of activity

As noted, modern societies structurally are the result of a continual division of the labor oriented toward the satisfaction of basic and complex human needs. This labor process involves *production*, the overarching process, which inherently includes *consumption*, *exchange*, and *distribution* as correlative processes, each of which can be understood as a productive activity (Engeström, 1987). This underpinning of cultural-historical activity theory implies that we cannot understand productive activity without also understanding its orientation toward the consumers of its products, the processes by means of which the products come to be exchanged and accumulated within society as a whole. As part of the environmental activist movement, Davie and his peers do not just produce any representation but they orient what they produce towards the anticipated visitors to the open-house event. The completed worksheet from their mathematics class, however, ends up in the garbage can. What most students correctly realize is that the only thing that really counts are evaluated pieces of work that contribute to their overall grades and grade reports. Though not generally included in the current (mathematics) education literature, this four-fold nature of productive activity allows us to understand why students and teachers are oriented towards grading, grades, and grade reports rather than towards mathematical knowledge and understanding. This instrumental focus on grades is the direct result of a capitalist exchange economy, in part held up by schooling, one of the constituent activity systems of Western societies.

The triangular representation of an activity system (Figure 2) expresses the mediated nature of activity. One can also say that the triangle exemplifies the distributed and societal nature of the knowledge that is required by and goes into the production of goods. That is, the representation suggests that we cannot attribute some product independent of the activity generally and its constitutive moments particularly. In a very strong sense, therefore, we may not attribute the product to the subject, who is only one of the *irreducible* moments of activity. We have to attribute it to the totality of the concretely realized activity, itself understood as an unfolding event. This totality involves the tools, division of labor, rules, and community. Thus, for example, if the mathematics teacher had encouraged his students to use the readily available computers and a graphing program, what Davie, Jamie, and Stevie produced would have looked very different. Rather than asking the teacher how to make a scatter plot, Davie simply could have chosen the scatter plot option from the menu. In this way, the use of computers would have realized schooling in a different way, leading to a more advanced activity system; and the difference between the pencil-and-paper- and computer-using systems would be theorized as a *tertiary* contradiction. In fact, the very problems observed in Vignette 1 may have been caused by the technology that the *rules* allowed, i.e. paper-and-pencil, which may have disabled Davie; this disabling relation between Davie and the tool is theorized as a *secondary* contradiction.

By contrast, during the environmental activism, the division of labor, available tools, and surrounding community were enabling, which allowed Davie to participate competently in mathematical practices while transforming the creek into a variety of representations. In the context of schooling, provided with a task the goal of which is not at all evident to him, Davie behaves such that the label “learning disabled” easily can be attributed to him. In the context of environmentalism, Davie produces something for the local society, as he engages in the knowledge exchange opportunities during the open-house event. Activity theory forces us to consider these products as the results of systems rather than those of the individual subjects that are only moment of the system as a whole. Moreover, as he recognizes at the end of the unit, the adults in the community learned *from him*, which means, he recognized the differential distribution of knowledge pertaining to the creek, as a result of which he becomes a knowledge provider.

The foregoing description makes immediately clear the ways in which current mainstream epistemologies differ from cultural-historical activity theory when it comes to locating cognition and learning. In constructivism, *individuals* are the seats of mathematical knowledge, which they construct for themselves and test for viability in the world (von Glasersfeld, 1987). Social constructivists do not differ significantly, as they assume that knowledge, once extra-psychologically constructed with others subsequently is constructed intra-psychologically by individuals (e.g., Cobb, Yackel, & Wood, 1992). In cultural-historical activity theory, on the other hand, any product is the result of the activity system as a whole and, therefore, is marked by all its constitutive dimensions. The fact that the practice-generating dispositions of any subject are formed in and through participation only emphasizes the importance of *activity* as the relevant unit and minimal category of analysis.

In any activity, the actions performed by the (individual, collective) subject are oriented towards some material object, which, through a series of actions, is transformed into the product. It is evident that not any action will do. Actions are a function of the ultimate product, already *envisioned* at the beginning of the work process. That is, whereas the object exists in material form, and is *ideally* reflected in consciousness, the future product initially exists only in ideal form. It constitutes the (collective) motive. Thus, cultural-historical activity theorists understand activity to be oriented to (material, ideal) object and (ideal) motive simultaneously. We may therefore speak of the *object/motive* of activity, an expression that makes salient this dual orientation of activity toward the sensual-material and ideal world. In fact, activity has to be understood in terms of its simultaneous existence at the sensuous-material and ideal levels (Leontjew, 1982). This category allows us to understand that there are considerable differences between the practical work of mathematicians and those of individuals who use mathematics as *means* for transforming their material objects into products. Thus, for example, the water technician in the Vignette 2 has as her object the creek, which she envisions (motive) to transform into a healthy feature of this part of the world. Similarly, our ethnographic study in a fish hatchery showed that the fish culturists did not consider themselves doing mathematics, despite the fact that there were many mathematical representations used and even mathematical modeling of fish populations was occurring, but rather considered themselves as raising fish; they used mathematics as a *means* in their productive labor to ascertain a healthy brood. Both technician and fish culturists were adamant about the differences between what they were doing and mathematics. The forms of consciousness associated with these activities are very different. The forms of consciousness with respect to the graphs that Davie, Jamie, and Stevie (not) produce are very different when they participate in normal schooling, where it is the object, versus when they are part of environmental activism, where these graphs are but means in/of their communicative efforts.

2.3 Society as network of activities

As a result of the progressive division of labor, modern societies have evolved into highly variegated networks of activities that are linked (a) by the exchange-related movement of products and (b) by the movement of persons. Figure 2 sketches some of the activity systems involved in the production of bread, which, as the final stage of production, steps outside of it to become the object of individual need satisfaction. In the overall process, the production process is distributed across the contributing processes of tool manufacture, seed production, harvest, and bakery. Second, individuals participate, in the course of their everyday lives, in multiple activities, for example, working as research scientists or laboratory assistants during the day, being shoppers in the bakery in the evening, and engaging in family life at night. The figure also allows us to understand that what Davie produces in the mathematics class is not part of these networks: it ends up in the garbage can. At other times, the results of such a task may lead to the production of a grade, which is recorded and included with other forms of evaluation, in a summary grade at the end of the school year. These grades and the associated diplomas are the actual material productions of schools. The activity in which Jamie and Davie participate while being in mathematics class is schooling rather than doing mathematics.

Davie does participate in a different activity system as part of his science curriculum: doing environmentalism. This activity *transforms* not only the world but also the social world, not in the least because of events such as the open-house activity. As a subject in this activity system, he contributes to producing mathematical representations that are exchanged and enter other activity systems that constitute society as a whole. The visitors to the open-house event are the "consumers" of the representations. Davie does not do mathematics in the strict sense, however. Instead, he uses mathematical representations to do what he intends to do: representing aspects of Hagan Creek to the general public that attends the open-house event, and those who read associated reports online and in the local newspaper where the results of the children's work are presented.

There is further indication as to the appropriate unit for the analysis of any material praxis that involves mathematics. Thus, to access their jobs, scientists and technicians require diplomas that they obtain in formal schooling (Figure 2). The production of these diplomas is an outcome of, and therefore characterizes, formal schooling. These diplomas are the products that travel into other systems, opening the doors to potential employees independent of what they may actually know. There is evidence to show that not what someone has learned and knows in mathematics and science comes to count in accessing further education and career but the grades received in the courses that they had taken. That grades rather than knowledge or competent practice matter can also be seen from the fact that some students "cheat" on mathematics examination, for it is only the grade that counts rather than the knowledge that the examination is supposed to measure.

Society continually evolves, which, because of the constitutive nature of its moments, implies that the relations between the activities and all the activities continually change. In a very strong sense, therefore, we have to take into account the history of the activity system and that of society as a whole. As a result of a five-year ethnographic effort in a fish hatchery, we showed that the current observable practices, including those that involved mathematical representations, were strongly related to the 30-year institutional history generally and to the 120-year history of fish hatching in the area more specifically. Thus, we found specific mathematical representations that did not exist in other hatcheries, which was a function of the work processes within the key hatchery's history; and we found younger employees using mathematical representations to model fish populations, whereas the older staff members unfamiliar with computers did not use these. That is, the mathematical performances we observed were mediated culturally and historically, whereby the latter had to be considered at different institutional levels, that is, the history of the field and society, the history of the particular institution, and the personal history of the people.

2.4 Subjectification and personality

Davie is not just a subject in schooling activity but also is part of a family, a shopper, and a participant in environmentalism. The cultural-historical activity theoretic framework presented here allows us to understand Davie as a person simultaneously characterized by very general and highly particular features. Thus, as participant in schooling, doing environmentalism, and shopping, he is a constitutive subject in the associated activity systems. But because the nature of the subject is a function of the activity system, the different subject positions that the person occupies are not equivalent. The longer Davie participates, the more familiar he becomes with the attendant activity as a whole, the more competent his practices become, and the more knowledgeability is associated with his participation. This process of developing as a subject in a particular activity system is denoted by the term *subjectification*. Drawing on ideas from the philosophy of politics, I understand subjectification as "the production . . . of a body and a capacity for enunciation not previously identifiable within a given field of experience" (Rancière, 1995, p. 59). The identification of this body, which occurs "through a series of actions" in and by this body – here those that Davie performs as part of the praxis that leads to the product of activity – "is thus part of the reconfiguration of the field of experience" (p. 59). As the subject of any activity system, Davie also is subject to and subjected to the same field of experience. This leads to the fact that he develops the same practice-generating dispositions modified by minor variations that are characteristic of the phenomenon. That is, as the subject of the activity, Davie takes on highly shared, societal features. He is recognizable, in his forms of participation, *as* student, environmentalist, or shopper. That is, as a participant in each of these activities, Davie realizes the corresponding object/motives, which are common to all corresponding subjects; and, as a participant, the subject Davie also changes together with all the other moments of activity. This is so, not in the least, because the activity, as a societal field, is constitutive of the habitus that generates the practices characteristic of activity.

As any person, who Davie is exceeds the shared, collective features that allow us to recognize him as a participant in a particular activity. But there is something highly singular about him that allows us to distinguish Davie from all other people who engage in the same and similar activities, including his classmates. In cultural-historical activity theory, *personality* is used to understand the integration of the societal features into a unique hierarchical "knot-work" of collective features (Leontjew, 1982). These "'knots' that unite the individual activities are not gathered up as the effects of the subject's biological and mental capacities that lie within him but are created in the system of relations that the subject enters" (p. 178). The knot-work that we denote by name "Davie," therefore, is a partial image of society as a whole, including, most specifically, all those activities that Davie is a member of. Although the individual moments of the resulting hierarchical knot-work are societal through and through, the knot-work as a whole is highly individual (singular), differing from other knot-works by the relative position of and bonding strengths between the contributing object/motives. Thus, for example, for Davie the object/motive of society-transforming environmentalism is dominant at the time of the science unit. The dominance of this object/motive leads to his participation in preparing another teacher to teach this unit, in the supervision of his same-age peers in another class, and to his participation in the open-house event, where he teaches adults and young children alike.

Davie is a working-class student, for he shares with his working-class family and peers a large number of object/motives. This is so because as part of his life in family and peer group, he participates in those activities that are also characteristic of the others; and he tends to value highly those object/motives that also are valued within his inner circle of family, friends, and acquaintances. Thus, schooling generally is a less-favored activity, and he does not subscribe to its object/motive (grades, diplomas) to the same extent that his classmates from bourgeois families do. Among the school subjects, mathematics in particular played a lesser role so that it mattered fairly little to him how well he did. Among the farmers and workers that inhabit the semi-rural municipality, doing mathematics often receives a much lesser esteem than most other activities. In fact, as shown by those families who live on my street, even working-class people do quite well for themselves without knowing any mathematics at all. More specifically, our research among local craftspeople, technicians, and sailors shows that the mathematics they learn in school and college is all but irrelevant in and to their working places for which the formal institutions are supposed to prepare them (e.g., Roth, 2012). For example, the environmentalist who worked as a water technician at a farm alongside the creek learned to handle very different mathematical representations and even produced some of them on her own. Yet she had found the college mathematics courses quite annoying and useless. There is therefore little incentive within the culture that would emphasize the need for participation in mathematical activity or the use of mathematics as a means of production. The forms of participation described in the opening vignettes are therefore not merely expressions of some biological disorder but are phenomena mediated through and through by the structure of the schooling activity and the manner in which it is knot-worked to the other activities.

Davie's way of participating in schooling leads to the production of a final grade report and diploma that only offer him possibilities to become a worker. In and through his participation in schooling, he therefore has contributed to reproducing class society. Although activity inherently offers the possibility of transforming the world, schooling actually stabilizes society rather than changing it into a more equitable one. That is, society changes because it is living. But, in the same way as the human body, society changes slowly leaving intact much of its structural features. Cultural-historical activity theory actually allows us to understand what is happening in terms of change: both the tendencies of transforming reproduction of the world and the reproductive transformation of the world. Mathematics, as means of production and object, may contribute to either process: making this a better society or reproducing it with all its inequities.

3 Opportunities Arriving from the Cultural-Historical Activity Theoretic Framework

Cultural-historical activity theory, as sketched out here, provides us with a means of (a) conducting research in mathematics education and (b) understanding mathematical cognition and learning in ways that other current theories do not. In my earlier work, I had focused, as others do, on individual cognition as if it could be reduced to itself and abstracted from everything else. In the late 1990s, I began an extensive ethnographic study in inner-city schools of Philadelphia, USA (e.g., Tobin & Roth, 2006). Most of the students attending these schools, predominantly African American and from the poorest parts of the city, would end up during their later life in the same conditions that they grew up in.

As a result of my research, I came to understand that in these schools the iniquitous nature of society is reproduced; I also realized that this reproduction occurs in and through the forms of participation in schooling practiced in these institutions. The fault could be placed neither on students or schools nor on schools and students; rather, the reproduction is the result of a dialectical negation that students and schools produce and that hinders students and schools in achieving their transformative potential. It would have been easy to relegate the reproduction to the school level, as sociologists tend to do it. But this is not the whole answer, because forms of participation also reproduce social class even when there are mixtures of working- and middle-class students. This means that the societal reproduction, a macro-level phenomenon, occurs in and through the everyday face-to-face encounters and relations that are reproduced and transformed at the classroom level.

Whereas much sociological research tends to focus on the macro-issues independent of everyday face-to-face work and psychological research tends to focus on interactions and individual engagement independent of more encompassing levels, cultural-historical activity theory *explicitly* theorizes the constitutive relations that exist between the very micro-level of practice and the very macro-levels, as well as every level that we may find in between. It therefore is a framework for researching and understanding how in everyday relations in the mathematics classroom, macro-level phenomena such as societal structure are reproduced. Most notably, this occurs because mathematics tends to be taken as an indicator subject, so that having succeeded well in specific mathematics courses becomes a criterion of entry to particular university programs. Whereas this is known, little studied are the everyday ways of doing mathematics lessons that allow more working-class students to fail more frequently than middle-class students. A cultural-historical activity theoretic perspective, however, might encourage mathematics education researchers to study questions such as how the emergent personalities, with differential emphases given to mathematics as object/motive, mediate the participation in specific mathematics lessons. Such studies require detailed documentation of individuals, the institutions they work in, and the history of schooling in a particular jurisdiction.

Another area of interest to mathematics educators might be the role and importance of “basic skills.” In back-to-basics movement, it was assumed that there are basic skills that an individual needed to exhibit to do mathematics. Evidence shows, however, that an improvement of “basic skills,” especially among the poorly served students of the US, did not lead to higher levels of knowledgeability that was required for university entrance. In fact, there is no evidence to support claims that the skills for requiring doing a long-hand division are the same that are required for doing a division using a slide rule, hand-held calculator, or computer (Roth, 2008). This contention makes sense within the cultural-historical approach, which emphasizes the mediational role that tools and their history in a changing society have on shaping the products of activity. Instead of thinking about the subject as the seat of knowledge, mathematics educators are encouraged to view the system as the bearer of knowing and the individual subject as constitutive moment. The form of consciousness characteristic of the activity is a function of the tools, and the operations called upon are a function of the actions, which differ when previous skills and actions come to be crystallized in the new means of production. This requires new forms of actions and skills, just as the spreading of cars required new actions and skills and decreased/abolished the need for proficiencies in walking and horse riding. Using the cultural-historical approach, therefore, allows asking questions about the role of particular curriculum elements that are currently assumed to contribute to mathematical knowledge without actual proofs existing to show that these presuppositions bear out. Thus, in very practical terms, we might ask about the need to include curricular topics such as longhand division, factoring of polynomials, or apply exponent laws?

Cultural-historical activity theory also constitutes an appropriate framework for asking questions about what we teach and how we teach it in mathematics classes. Thus, for example, those mathematics educators with ethnomathematical interests tend to report findings about how mathematical representations are used or show up in a variety of often-mundane practices (e.g., Pinxton & François, 2011). Those advocating classical forms of mathematics study how students engage with mathematical objects. The cultural-historical approach, as discussed above, makes evident that the forms of consciousness and knowledgeability in the two approaches are very different. Those who use mathematics as means in the pursuit of non-mathematical object/motives relate in very different ways to the field than those whose object/motive are mathematical objects transformed into other mathematical objects for the purpose of producing mathematical knowledge. The question, then, is not one of different epistemologies – our practical epistemology allows us to understand the differences that will result when the object/motives differ.

Most importantly, in the context of this special issue, cultural-historical activity theory provides a framework for studying the change of mathematical teaching and learning in the context of a critique of bourgeois society and the societal changes that arise from such critique. This is so because the theory has all the characteristics that are required. First, because schooling is an integral and constitutive part of a network of activities that we denote by the term society, changes in society mean changes in the entire network of relations and within the individual nodes (activities). This is so simply because of a whole-part relation between society and individual activities – easily modeled, for example, in artificial neural networks or constraint satisfaction networks. Here, changes in one relation or one node ripple through the entire network, and changes at the network level inherently mean changes in the individual nodes and links.

Second, cultural-historical activity theory explicitly links the structure of society and its moments to history, that is, it understands society as a living phenomenon. When properly using the theory, researchers inherently attend to the structural (synchronic) and temporal (diachronic) nature of this human life form. It is inconsistent with the theory to study students like Davie by giving them a psychological test away from the societal and structural resources that normally characterize his life and label him "learning disabled," with grave consequences for what happens to him and his developmental possibilities. The theory forces us to study the synchronic and diachronic relations that end up being grossly attributed to individuals. Societal changes are not the result of outside forces but emerge from the society itself much as a (a) meeting that begins as an expert think-aloud session over mathematical representations may turn into a tutoring session as a result of the practical relations and activity and (b) frustrated and failure-laden engagement in a mathematical task may turn into enjoyment and success. Consequently, cultural-historical activity theory is an ideal framework for understanding how the structural properties of society not only tend to be reproduced but also offer opportunities for overcoming existing societal barriers instituted by achievement on mathematics tests. It offers a way of understanding the person as a unique "knot-work" of societal object/motives. And, perhaps of the greatest importance of all reasons, the cultural-historical approach allows us to understand, theorize, and promote the role of mathematics education in the self-transformation of society into one that is consistent with its democratic ideals rather than one characterized by the capitalist market exploitation at the expense of the most vulnerable members of society.

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Should Learning (Mathematics) at School Aim at Knowledge or at Competences?[1]

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Abstract: The Czech school reform tends to aim at competences rather than at knowledge. We attempt to bring some evidence that the *mathematical competences* is a vague term without precise scientific as well as practically useful contents. Based on the analysis of the Czech data from PISA 2003, we could not substantiate the differences in teaching practices referred by students with different mathematics test results, i.e. allegedly with different levels of math competences. Thus, any more or less effective competence-teaching practices could not be identified. Similarly, in TIMSS 2007, we did not find any differences in teaching practices between the Czech "pro-reform" and "antireform" teachers.

We further investigated the reformists' requirement to set up the real life, on the one hand, a starting point of the school teaching/learning, and, on the other hand, its ultimate goal. In the studies exploring the issue, we found no definite results in favour of the use of real-world problems in school mathematics. Similarly, the effectiveness of any particular method or practice of school knowledge acquisition on a far transfer into the everyday or even professional life cannot be proved. Moreover, the usefulness of the concept of transfer itself was questioned in recent debates as redundant (existing besides the concept of learning).

We conclude that the Czech reformist discourse is predominantly based on extracting partial moments of the teaching/learning process, putting them as antagonisms and labelling them as universally good or bad. On the contrary, we argue that these different moments, however opposite they could seem, stand as mutually linked parts in the real teaching/learning process.

Keywords: knowledge, competence, teaching/learning process, real-world problems, transfer, school reform, reformist discourse

For more than twenty years, the educational discourse in the Czech Republic has been engaging in debates – sometimes implicit – focusing on an issue which can be summarized as follows: is the child's learning more efficient if he/she appropriates items of knowledge (mostly identified with memorizing and drill) or if he/she constructs, discovers, leads inquiries or solves problems? The first orientation tends to be termed "passive", whereas the second "active". This controversy appears in educational practice and theory under the label of *traditional* teaching/learning *or* teaching/learning based on *direct instruction* vs. *constructivist, competence* or *skills* oriented teaching/learning.

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The reformist discourse in the Czech Republic, as in many other countries,[2] advocates the latter position. In striving to answer the question of effective teaching/learning, it has set up a stark opposition between these different approaches to teaching. The experts of the National Institute of Education, for instance, express the crux of the curricular reform (and, more generally, the reform of teaching/learning) in the following, succinct way: "Simply put, students should no longer be required to learn heaps of encyclopaedic information. Instead, they should learn to think logically, search for information and use it. They should be encouraged to be more active in class and to deal with problems independently. … Graduates should not only become experts in the given area but should also acquire the so-called key competences – the ability to communicate, co-operate, bear responsibility, make themselves understood in a foreign language, use information technology. Furthermore, they should be willing to pursue life-long learning. This will be essential in their future lives, as they will be required to respond to social and technological changes." (Franklová, 2009, p. 22)

This somewhat Manichean treatment of the problem results in a series of antinomies which, however, are not evidence-based. In the present article, we will describe and discuss in greater detail the (school) knowledge vs. competence (for life) controversy and the extent to which such an approach can be substantiated by the research evidence in the field of education.

Actually, this debate revolves around the following issues:

(1) With an appeal to the decline of the results shown by Czech students in PISA and TIMSS surveys, reformists allege that school teaching/learning (including the teaching/learning of mathematics) is inefficient. School teaching/learning should focus on the *development of cognitive competences* (*of thinking skills*) instead of *subject matter knowledge*, they say.

(2) The PISA surveys assess *competences for life*. Therefore, it is said, we must set *real world-problems* – instead of de-contextualized "abstract" knowledge – as the starting point of the teaching/learning.

(3) They believe school teaching/learning focused on competences to be more efficient since it allows for a long-term (far) *transfer* – which is meant to be in line with the alleged general long term aim of school teaching/learning: the practical application of its outcomes in life.

The question that needs to be raised is the following: which approach or method of teaching/learning is preferable? Is a general answer to such a question available at all – irrespective of the cognitive level of students and the potential of their proximal de-

[2] The benefit knowledge acquisition may have for the intellectual development of the student is periodically questioned in many countries. The most marked example is the USA, where blanket evaluations of students' school results show that despite the improvement in basic competences, the students still do badly in higher learning processes. This leads to repeated criticism levelled at the school for its not being able to ensure that students will acquire the methods and competences related to thinking skills. The teachers are subsequently pressed to focus their teaching on thinking competences (de Bono, 1976; Ennis, 1987; Paul, 1993). The implicit opinion is commonly shared that the school puts too much emphasis on knowledge acquisition, while it should preferably pursue the development of intellectual competences applicable in practice.

velopment, or regardless of domain of learning or of the nature of the task? And what kind of educational research evidence is supposed to help us in such a deliberation?

1 Are Competences the Goal to Which the Curriculum is Just a Tool?

From the reformists' point of view of the education, the curriculum is a subject serving other, more primary objectives. Within the Czech curricular reform, its role is "subservient" or instrumental because students' competences are the goals. This is apparent from the wording of the policy reform document called *Framework Education Programme for Basic Education* (RVP ZV, 2005) and it is even more obvious in its further interpretation. For instance, J. Maňák states that the reform consists in a radical change of objectives and of the content of education: "Elementary education should help the students to gradually develop key competences and provide a reliable basic general education, oriented especially on the situations close to life and practical acting. These are important target values not previously delimited in such a pregnant and univocal way. … The curriculum is understood as a tool that has to lead to the achievement of key competences" (Maňák, Janík, & Švec, 2008, p. 35). The subordinate role of the curriculum is further revealed in the structure of the text of the RVP ZV, which puts forward nine very broad educational areas followed by educational fields. Only in the last step attention is paid to school subjects and their curricula. The document relates these levels within a hierarchic order: particular knowledge is included by way of curricular *example* only after target competences in the educational area, field and subject have been delimited.

The quite indefinite term competence (construed as a "system of knowledge, skills, abilities, attitudes and values that are important to the individual's personal development and to the individual's role in society", RVP ZV, 2005, p. 14) comes, as Ropé and Tanguy (1994) show, from the sphere of vocational (professional) education. The conception and instruments of vocational education are meant to provide the model for all education, from pre-school to university level. Laval (2004) explains that the term "competence" is neutral only in appearance: it seemingly expresses a universal psychological construct relating only to an individual and his/her acquired, largely implicit qualifications, not controllable by the school. This, however, obscures the fact that it involves a very unclear connection between theoretical knowledge and skills on the one hand and of practice on the other. Laval thus reminds us that with the introduction of *competences*, we are led into the world of evaluation tools, control, supervision and of a gradual search for maximum rationality. But this world is the world of out-of school, especially of work, in which a set of required processes and skills needed to fulfil particular operations is established and adapted ad hoc to meet the criteria of profitability. The competence is ultimately destined to serve a practical, utilitarian purpose. Hence, at the heart of the term competence, one finds the criterion of being "close to life and to practice".

Nevertheless, the criterion of utility for practical life is gaining ground even within the critique of the school form of teaching/learning. For instance, the reformists' critique extends even to the teaching of mathematics which, in its "traditional" form, allegedly ignores the natural, everyday life of the child. They believe this serves to undermine students' motivation: "After all, as early as the first class, the renaming of the subject itself from "Sums" to "Mathematics" and the compulsory use of the terms "plus" and "minus" instead of the until then common terms "and" and "without" bears witness to the tendency to ignore the natural connection of the child to his/her concrete world" (Mleziva, 2011, p. 5).

Yet PISA itself states that the focus of mathematical tasks was on the "functional use of knowledge in solving real-life problems, rather than on ascertaining to what degree they had mastered their studies of formal mathematics or the degree to which they were facile with particular facts or procedures" (*Learning Mathematics for Life*, 2009, p. 36). And in defining mathematical literacy, the ability of students to solve real-world problems with the use of skills and knowledge gained both through school teaching and in everyday life is posited as a criterion of successful teaching. Competences should thus serve both as the starting point and as the objective of teaching: "Mathematical literacy is about dealing with 'real' problems. That means that these problems are typically placed in some kind of a 'situation'. In short, the students have to 'solve' a real world problem requiring them to use the skills and competences they have acquired through schooling and life experiences." (*Learning Mathematics for Life*, 2009, p. 20)

The data of both PISA, and partially also of TIMSS, may therefore be expected to give us a more precise idea of the contents of the term "competence" and of the kinds of teaching that influence – positively or negatively – the development of such competences.

Czech reformists often refer to the PISA and TIMSS surveys in two different ways. On the one hand, they, in accordance with PISA, emphasize that the objective of the reform consists in the development of "competences for life instead of knowledge". On the other, they explain the bad results of the last surveys in the following way: Czech students lag behind because the reform has not been implemented consistently and has not been carried out fully yet.

Our question is whether it is possible to identify in these surveys any differences in Czech teachers' ways of teaching and whether some of these ways lead to better student results than others.

During the TIMSS 2007 survey, Czech teachers were, among other things, asked to answer the following question: "To what extent do you agree with the following judgements concerning the reform in progress: a) I support the reform; b) the implementation of the school educational programmes[3] (SEP) is just an outward change?" This allowed us to construe a group of "pro-reform" teachers that agree ("definitely" or "rather") with the first statement and disagree with the second,

[3] The school educational programmes should be a concrete tool by means of which the curricular reform privileging competences is realized in the given school.

and a group of "anti-reform" teachers who answered in the opposite way. On this understanding, 15.1% of the teachers in question were in favour of the reform, 61.5% of them were against.

When comparing the groups of teachers, we focused on whether there were any differences in their students' overall results in the mathematics test. The difference in the national Rasch score (mean 150, STD 10) was 2.47 in favour of the pro-reform teachers. But even this small difference could be attributed to the fact that a disproportionate percentage of the pro-reform teachers from the Czech sample teach at eight-year grammar schools where the performances in the test are generally higher in comparison with the corresponding grades at basic schools. When a separate comparison was made between pro-reform and anti-reform teachers first in secondary basic schools and then in eight-year grammar schools, the differences in results were lower than 0.5 point.

Furthermore, we compared these teachers based on the methods of teaching they use in their classes.[4] When we compared their answers, some differences became apparent. The pro-reform teachers purportedly frequently ask their students "to apply facts, concepts and procedures to solve routine problems", "to explain their answers", "to work together in small groups" and, on the other hand, do not often require them "to relate what they are learning in mathematics to their daily lives". Even though some of the differences could seem surprising, they were immediately challenged when the same comparison was made based on the answers of students. From this point of view the pro-reform teachers differ from the anti-reform mostly in that in class, students "use calculators", "begin their homework in class" and "explain their answers" more often, and that, on the other hand, they "practice adding, subtracting, multiplying, and dividing without using a calculator", "review their homework" and "work on fractions and decimals" less often. We can see that there is very little overlap between the answers of teachers and students – they agree only in that students are required to explain their answers more in classes taught by pro-reform teachers.

We also wanted to find out, whether particular answers given to question 17 (teacher questionnaire – TQ) and to question 10 (student questionnaire – SQ) influence the overall performance in the mathematic test in any given way. At first, we calculated the correlations (Spearman's rho) with the overall result for individual items. The correlations were very low or negligible.

We also examined whether a certain combination of the items had a stronger influence on test results than individual items. We carried out the factor analysis individually for the items in question 17 (TQ) and in question 10 (SQ). For both of the questions, a two factor model was suitable for the description of the connections between the items. Moreover, for both of the questions, only one of the factors was saturated by the differences between "pro-reform vs. anti-reform teachers". The

4 Teacher questionnaire – question 17: "In teaching mathematics to the students in the TIMSS class, how often do you usually ask them to do the following?" (items *a–l*)
Student questionnaire – question 10: "How often do you do these things in your mathematics lessons?" (items *a–q*)

contents of most of the other items saturating any of the factors allowed for the hypothesis that these are the factors characterising, to some extent, the reform and, on the other hand, traditional teaching. So we used this result to create the aggregation of items in both questions 17 and 10. But even the correlations between the aggregated items and the overall mathematics test results were negligible (the highest was $r = 0.085$).

On the other hand, we found an interesting correlation of the test results with the school grade in mathematics ($r = 0.605$). Evaluation using grades is often criticised by the reformists as inadequate and as a formal expression of success at school that leads to undesirable types of motivation for the children and to the acquisition of knowledge which is merely formal and precludes deeper understanding.

In the PISA 2003, we were not given the possibility of comparing two predefined groups of teachers because no questionnaire for teachers was administered in this survey. Instead of this, we started with the assumption that the test result itself is an expression of competences and that the ways of teaching that would significantly correlate with the test results could be considered as pertinent in shaping such competences.

The question Q38 of the students' questionnaire does not give a detailed account of ways of teaching in class, unlike the TIMSS 2007. Nevertheless, for some of the items it could be assumed that they may be connected with the reformist mode of teaching and thus influence the results of the test.[5]

Similarly to the TIMSS, we found negligible correlations between the answers to the question Q38 and the overall test results and a much higher correlation of the item 39a, which asked about the mathematics school grade in the most recent school report ($r = 0.425$).

The correlation of the grade and the PISA test result is thus lower than for the TIMSS test. But it is questionable whether it confirms that there should be an important difference between the nature of the TIMSS and PISA tests, or, whether the difference corresponds to the difference between mathematics knowledge and *mathematics* competences.[6]

Whether we consider the result of the test to be an indicator of the level of knowledge or of the level of competences, we come to the conclusion that, bearing on the data of PISA and TIMSS, there is nothing to deduce about the efficiency of ways of teaching. It also is not clear from the data which ways of teaching allow us to distinguish between pro-reform and anti-reform teachers, or between teaching of competences and the traditional teaching of knowledge.

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5 Student questionnaire – *Q38*: How often do these things happen in your <Mathematics> lessons?

6 The question arises here to what extent the statement that PISA assess the ability of students "to use their mathematical knowledge in solving mathematical situations presented in a variety of settings" (Learning Mathematics for Life, 2009, p. 36) is valid and to what extent focus shifts from solving mathematical situations to the quality of the description of the context, the frequency at which the student comes in contact with that context and its proximity or distance from the school context. The authors underline that the ability to switch between the school context of using mathematics and its use in various everyday contexts is taught – to a different extent in various countries.

2 "Real-World Problems" (RWP) as a Starting Point of Learning

If the competences are delimited at all, their key feature should consist in their relation to real life, in the two following senses:

1. contexts of everyday life should form the starting point of learning,
2. the general long term objective of school learning should be the use of its results in practice, in real life – this presupposes their transfer far in time and context.

In this section, we will discuss the first issue. The question of transfer will be discussed in section 3.

The popularity of teaching based on everyday life problems was augmented by the development of the so called situated learning (Lave & Wenger, 1991). The situated learning approach had been initiated by the justified criticism of the application of intelligence tests coming from the western civilization area to people from other cultures. This is because successful solving of tasks in these tests requires ways of thinking and knowledge gained by systematic school learning. But when these were applied to populations with significantly different cultural background or to culturally disadvantaged (ethnic minorities in the USA or indigenous populations in Africa), their results were very poor. This led researchers to the idea of monitoring the skills of these people in their everyday life. The positive result was the discovery that the tested people are not intellectually retarded, because in situations that are familiar to them they master the required cognitive operations and skills (Cole, Scribner, 1981).

Significantly, the situated learning translates into French as 'learning in context' (*apprentissage en contexte*), a somehow inaccurate expression, but one which makes explicit reference to an important dimension of situated learning, that of context, which in turn reminds us of the other necessary term of the relation, "text", making salient the "text–context" relationship. This turn towards situated learning, towards forms of cognition and learning in practical situations (e.g., intellectual operations developed by tailors apprentices in Jean Lave's experiments in Liberia, 1977), towards learning in practice (e.g., everyday arithmetic in the research conducted by Scribner, 1986, and by Rogoff, 1990) led to a full appreciation of cognition as a set of cultural practices and of the impact of the context on learning. And the term "apprenticeship" was established to denote the learning embedded in a (real-life) situation (contrary to the term "learning" denoting the formal school learning).

At the same time, it may have led to the overestimation of this form of learning at the expense of the importance and function of school forms of cognition and learning, of what we could call the "text in the situation". In conjunction with the reviving educational reformism and a return to student-centeredness, this led to the overall negation of the developmental significance of school forms of cognition. Situated learning in contexts of practical life of the individual was placed on a pedestal, almost as a model for learning at school. Reformist (student-centred) conceptions of teaching/learning are strongly nurtured by this conception (Štech, 2008).

Many users of situated cognition theory are insensitive to the fact that learning at school is also learning in a context with its own specificities, a context which represents a community of practices largely derived from a concept of scientific knowledge. A comparison with extra-curricular contexts makes it evident that the objective of school is epistemic. It aims at the transformation of modes of thinking, of experiencing, and of the self. This requires a clear conception of the relations between spontaneous learning (the kind of learning we do, and what it is we learn, in everyday contexts) and education, formal learning and development. School teaching contributes to the so-called intellectualization of cognitive operations – to working with principles with economical abstraction, which brings a different quality of attention, memory or thinking and "raises" the spontaneous thinking in everyday practical situation to a higher level. Let us show the difference on one example.

Carraher and Schliemann (2002) refer to some examples of thinking of adults who did not attend school and to limitations of their knowledge gained in everyday practice:

"We asked Brazilian school children and street sellers who had received little or no instruction on multiplication to solve aloud pairs of verbal problems where they had to compute the price of a certain amount of chocolates based on unit prices (Schliemann, Araujo, Cassunde', Macedo, & Nice'as, 1998). The following is an example of the problems pairs we used: Type 1: A boy wants to buy chocolates. Each chocolate costs 50 cruzeiros. He wants to buy 3 chocolates. How much money does he need? Type 2: Another boy wants to buy a type of chocolate that costs 3 cruzeiros each. He wants to buy 50 chocolates. How much money does he need?

Participants first solved a problem where the larger number denoted the price of one item and the smaller one indicated the number of items to be bought. Immediately after they were given the corresponding problem where the smaller number denoted price and the larger one denoted number of items and were asked whether they knew its answer without doing any computation. If they used the former problem to answer the latter, we took this as an indication that they relied on the commutative property of multiplication. The group of school children who had received school instruction in multiplication (second- and third-graders) solved the first problems in each pair via multiplication and frequently relied on the commutative property to answer the second problems. In contrast, street sellers tended to use repeated additions throughout and rarely invoked the commutative property to answer the second problem. Instead, they successively added the number of cruzeiros, a cumbersome procedure leading to frequent errors if they had to add, for instance, 3 cruzeiros 50 times.

The above results suggest that, although people can learn meaningful mathematical ideas in mundane, non-academic situations, they nonetheless need access to new symbolic systems and representations they are not likely to acquire out of school." (2002, p. 253–254)

The issue of effectiveness of the use of "real-world problems" (RWP) in school teaching/learning is also dealt with in the National Mathematics Advisory Panel

(NMAP) report. It states that, especially, this term is used in a very diverse way: "A serious problem in synthesizing the research in this area is that there is no clear, agreed-upon meaning for "real-world" problems" (NMAP, Report of the Task Group on Instructional Practices, 2008, p. 98). In comparable studies on contextualization, the use of "real-world contexts" in school problems proved effective, but only if these were contextualized in a similar way. Regrettably, the students did not manifest the improvement in other, more general skills (computation, simple word problems, and equation) beyond the solving particularly contextualized problems. Other studies, which unfortunately showed some methodological flaws, mostly found positive effect of learning on contextualized problems, but the effect was measured on RWP the control groups have not learned (NMAP, ibid, pp. 96–98). The effect of teaching based on RWP, therefore, does not reveal to be the consequence of the particularity of the problem originated in everyday life, but rather of the fact that the context taught in experimental groups was new, not commonly practised in the school and – paradoxically – unknown to the students of control groups in connection to mathematics.

These results are very similar to the results of studies on transfer we will mention in the following section. In most of the studies on various teaching practices that should increase the transfer, the impact is predominantly observed on the *near* transfer. The effect is found in contextually close domains and in relatively short term. Long-term transfer to problems with more different contexts was not proved.

The studies discussing the effect of RWP emphasize more the dragging the students into the context or into the story of the problem as a parameter of efficiency rather than using the elements of the non-school environment just as props. Regarding this, we doubt the PISA problems – given to the teachers in the Czech Republic as an example of "teaching for life" – meet this condition. At least, it is evident that it is not possible to set a problem that meets such a requirement universally for all the students of the given country.

The ambiguity of the conclusions of the studies about the effectivity of RWP rather corroborates the opinions of Son and Goldstone (2002). They report on three experiments on contextualization they conducted. In these experiments, the control groups always had better results in the tests than the experimental groups learning under the conditions of various types of contextualization. The authors then come to cautious conclusions:

"Our experiments suggest that it is not that concrete experiences, activities, and demonstrations are generally good or bad for transfer, but rather these manipulations cause particular construals that affect learning and transfer" (ibid, p. 75).

Our results should not be taken as opposing contextualization, personalization, or learner adapted approaches. Instead, our experiments show that the 'one size fits all' approach, where personalized contexts are simply grafted onto contents, could have negative cognitive consequences. Undoubtedly, participating in activities and evoking real world knowledge is influential and can result in effective activity-specific encoding. For example, warehouse drivers in a dairy organize information according to warehouse location and pallet size whereas consumers typically encode by general categories (Scribner, 1985). Their activities and perspectives allow them to selectively encode relevant information. However, this context-bound encoding leads to potential pitfalls. (…) A de-contextualized understanding may be beneficial for learning structural principles by leading to a broad, detached understanding of a situation rather than being guided by a particular perspective" (ibid, p. 76).

It appears, therefore, that to base education on problem tasks may have a certain importance at a given stage of teaching/learning or for a given sub-class of students. Nevertheless, their positive effect is by no means to be expected to apply universally and it makes no sense to understand the issue as that of two mutually exclusive types of approach.

May it be, though, that real world problems and the ability to handle them serve to prove whether the ultimate objective of teaching/learning has been achieved? Let us have a look at what research has to say about real world tasks as a criterion of the effectiveness of teaching/learning.

3 "Problems in the Real World" as the Objective of Learning

From the point of view of psychology, the ambitions that the school learning should be the "learning for life" corresponds to the issue of the far transfer – an issue studied for more than one hundred years. The classical concept of transfer has been based on the notion of identical elements: the transfer happens thanks to the correspondance or overlapping of elements contained in the learning contents and in the target situation. Cognitive revolution reworded this in the sense that the transfer happens when the mental representations of the original and the target situations are identical or overlapping (Lobato, 2006, p. 433).

This classic interpretation was criticised during the whole 20th century, so that some authors state even after a hundred of years the results of research are still ambiguous and the problem of transfer remains open (Barnett & Ceci, 2002; Wagner 2006; Lobato 2006).

Barnett and Ceci (2002) see the basic problem of unsatisfactory results in divergent interpretations of transfer by researchers. This makes comparability of their results rather impossible. Therefore, even though there are many studies on transfer, they infer it is not possible to come to more general conclusions.

That is why they suggest the taxonomy of transfer based on two global dimensions – that of content and that of context. The dimension of content is further divided according to the following criteria allowing to classify the transfer: 1. the skill to be learned, 2. the nature of the expected change, and 3. the requirements on the memory (whether there are prompts available and whether the nature of the process is more a recall or a retrieval). The dimension of context consists of six sub-dimensions: (1) the domain of knowledge, (2) physical context (e.g., in school vs. out of school), (3) time context (near vs. far transfer), (4) functional context (different from the physical one, but the distinction is not quite clear), (5) social context (learning alone or in cooperation), and (6) modality (oral or written, visual or auditive transfer, etc.).

They think about each of the sub-dimensions in two values (near – far transfer) defined as the distance between the learning situation and the situation which the learned skill should be transferred to. But even with this limitation, the articulation of context according to these dimensions leads to 64 combinations. Even after reducing the number of dimensions to the two or three most important ones, the authors found only very few studies (often just one) for every cell of combination matrix, so that it was not possible to make any comparison. But they were not even able to classify many of the studies according to their taxonomy, because these did not provide the necessary description of the methodology.

Barnett and Ceci (2002) consider their taxonomy to be, on the one hand, a tool that could measure the comparability of studies and on the other hand, to be an appeal to fill the empty cells, i.e., the missing combinations of context dimensions by further researches. Just as an example, they have not found any study testing the transfer to the far knowledge domains, far physical context and far time. This finding is a very interesting one as this is just the kind of transfer the reformists expect the school teaching/learning should guarantee.

Nevertheless, most of the authors criticizing the traditional research on transfer disagree with the way the transfer is conceptualized. It is a seemingly very broad criticism using many partial re-conceptualizations and also many new terms. But we still think that it is possible to summarize it into three basic issues.

1. Most of critics point out that it is necessary to *understand the transfer dynamically*, not statically. The acquired knowledge is not transferred to the new situation directly as a priori prepared, fixed and abstract schemes. In a new situation, the learner explores at first what from the previously acquired knowledge could be used to solve it. Thus, the knowledge changes in the process of transfer, adapts to the new situations and enter into new combinations: "… there is little evidence for some monolithic skill or piece of knowledge being carried over intact from a unique prior situation to the present one. On the contrary, the students are wrestling with multiple, competing ideas. (…) They do far more than deploy this knowledge. They draw upon it selectively to deal with the unique predicaments at hand. They have not simply unloaded a prior solution from their storehouse of knowledge. They have crafted it on the spot, adjusting and adapting their prior knowledge in the process" (Carraher, Schliemann, 2002, p. 19)[7]. From this point of view, it is impossible to determine in advance what will be considered as a transfer or as its criteria. But this is just the way the quantitative researches are conceived according to their randomized controlled trial (RCT) ideal as they attempt to measure transfer in terms of the improvement of the performance defined in advance.

[7] Similar description of the learning process could be found in the article by Wagner (2006).

On the other hand, qualitative studies do not start with an idea of the result of the predefined transfer, but make the content of the transfer their object of research. "… evidence for transfer … is found by scrutinizing a given activity for any indication of influence from previous activities and by examining how people appear to construe situations as similar using ethnographic methods, rather than relying upon statistical measures based on improved performance" (Lobato, 2006, p. 436). Also the final part of the Wagner's article can be, in fact, understood as an appeal to use such research procedures that are inherent to the ethnographic approach. Especially, he stresses the necessity to include the manner the student himself/herself sees and processes the problem into the researcher's analysis of the process of "transfer in pieces" (Wagner, 2006, p. 68).

Even though Lobato states that it is necessary to distinguish between the criticism of methodology in the transfer research and the criticism of its theoretical concept, it seems that this can hardly be achieved. The "new" concept of transfer clearly requires data that could be obtained only by the qualitative methodology. Lobato does not pay attention to this necessary requirement; in fact, she complies with it only verbally. Her actor-oriented approach – sometimes she even uses the unclear expression of actor-oriented transfer – is nothing else than the ethnographic approach to the research of transfer.

The qualitative methodology, on the other hand, clearly does not provide the data inherent to the experimental and quasi-experimental research. Does it make sense to prefer one of these to the detriment of the other?

2. The criticism is also aimed at the classic idea that the far and flexible *transfer* is ensured by *mastering abstract schemes* and concepts. In this vein, Wagner objects to such an assumption that, at least at the beginning of a learning process, no fixed mental representation is created and that there is nothing that could overlap and with which the new situation could be compared. We can see a certain logic in his opinion: to search for overlapping with just one concrete (previously created) mental representation and not with another one, the student would have to categorize the new situation (problem) under the more general category of similar situations. But the learner cannot be aware of such a category at the moment.

Nevertheless, this is true for the unguided problem-based learning only. However, in Wagner's discussions with Maria, one of the participants in his research on probability problems (Wagner, 2006), we can see something slightly different. In solving the problems asking for the most appropriate number of coin tosses under various conditions for winning, there was a great positive change in answers and comments after Maria learned about the law of large numbers and central limit theorem in school.

Apart from other things, this clearly shows how the classroom context of school subjects curriculum (the matter "we just go through") originates the awareness of the fact that the next problem to be solved probably belongs to the same class of problems. This generates the basic frame of inter-contextuality – still in external form that has to be internalized and further developed. Here, Wagner, but also others, set up a too sharp (antagonistic) contrast between the mental representation of a class of problems (schemes) and the mental representation of a single problem or situation which Wagner calls a collection of pieces of knowledge and particular schemes. The way Wagner describes the deepening of the understanding and what he considers to be the process that leads to the abstraction[8], could be described as building and deepening of the mental representation as well, the germ of which comes of the internalization of a scheme or algorithm provided by the teacher's direct instruction. Here, in fact, we face the old and well known problem: the tendency to excessively stress the difference between the bottom-up and top-down process.

Finally, the transfer is a matter of the process of understanding abstract knowledge (terms, algorithms, schemes). Understanding is never a one-shot event; it is a long-term process where the bottom-up and top-down approaches alternate and the understanding becomes gradually deepened.

3. The traditional research on transfer gets criticised for conceiving the context as the features of the problem situation perceived only from the point of view of the researcher, i.e., most often as *external parameters of the problem*. Therefore, these features are considered to be superficial and irrelevant to the way the situation (problem) is understood by the students. The opposite approaches emphasize the context as highly individual and therefore necessarily mediated by students' previous knowledge and experiences, which, moreover, vary inter-individually. We consider this to be an important argument corresponding, among others, to our doubts about the real-life nature of the PISA problems.

On the other hand, even most of the critics overlook that the school learning – as we already stated above – generates a specific context tending to be similar for the students in one classroom. This happens, among others, because their previous knowledge and experiences consist, maybe even dominantly, in knowledge and experiences acquired during the teaching/learning of the given subject. The context of the school class serves as a support for inter-contextuality – or as a "focusing phenomenon" (Lobato, 2006), or "framing" (Engle, 2006) or "context sensitivity" (Wagner, 2006).

Then, the student logically gets into problems if he or she is suddenly deprived of this support. In our longitudinal research (Rendl, unpublished data) we observed this difficulty in the learning within the school itself: students manifested some difficulties in applying mathematics knowledge, even well acquired, to other school subjects (physics, geography, chemistry). It is evident that the leaving of the contextual support of the school context represents a specific problem. At the same time, it is questionable if it is the very school that can completely solve the problem of helping the students to find the meaning of school knowledge and of its effective use in everyday or even professional contexts which, moreover, significantly vary between individuals.

8 "… abstraction was a *consequence* of transfer and the growth of understanding – not the cause of it" (ibid, p. 66).

To summarize the issue, the recent discussions about the transfer bring the following methodological and theoretical questions:

- Should we investigate the general conditions for improving the learning performance or explore the individual processes of qualitative changes during the learning process? Should we examine the consequences of learning in terms of the results of formalised activity within an experimental design or should we explore the process of learning that leads to the performance?
- Should we start from the "meaningless and rigid" abstraction moving to the activity in a concrete situation or start from the situated activity moving to the understood abstraction? From de-contextualization to contextualization or vice versa?
- Should we study the learning process as isolated from the surrounding context or should we analyze it as influenced by various context frames in their full scale – from classroom to family and to broader culture of student's life?

We think that these contradictions comprehend real moments of the learning process and, in consequence, they contribute to shape the design of the research on learning. However, as we demonstrated, it is not reasonable to postulate them as antagonistic instead of complementary.

As the evidence-based discussion over the issues mentioned above demonstrates, it is also more and more clear that transfer and learning are one and the same process and that one of these terms seems to be redundant. In this, we are close to the opinion of Carraher and Schliemann (2002): "Our goal is to recommend not an 'improved' version of transfer, but rather the abandonment altogether of 'transfer' as a view of how learning takes place" (p. 22) "… there are other ways to frame the way prior knowledge and experience contribute to learning" (ibid).

4 Conclusion

The Czech reform documents and their interpreters relegate learning of knowledge and postulate competences as the main objective of learning. But 'cognitive competences' appears to be a very vague term. If the term is not used simply as a general term for knowledge, skills, abilities, etc., its extension remains unclear and is used very arbitrarily.

Our effort to operationalize competences using the data of TIMSS 2007 and PISA 2003 was not successful. In the TIMSS data, we were unable to identify didactic procedures (teaching classroom practices) that would distinguish "pro-reform" teachers (allegedly trying to develop the students' competences) from those who are "anti-reform" (allegedly using inefficient traditional methods). Also, no differences in the mathematics test results were found in students taught by either of the above mentioned groups.

Similarly in PISA 2003 we identified no impact on the mathematics test results (said to measure the level of "competences for life") which could be ascribed to didactic procedures – and thereby no differences in their effectivity could be found.

According to the reformists, the school policy emphasizing competences should also be realized by means of learning based on the RWP. Unfortunately, we did not find any discussion of competences in the field of psychological research. Instead of competences, the analyzed studies explore the use of RWP from the point of view of mathematical knowledge and of the ability to apply it. Moreover, the results of these studies are ambiguous. If the effectiveness of learning based on RWP is higher, it is mostly when the measurements are made on problems contextualized similarly to the original problems. On the other hand, some studies report that excessive use of RWP in teaching/learning may even have negative effects: the student remains fixed to a particular conceptualization connected with a particular contextualization and he/she fails to form more general concepts, categories and principles. The question, therefore, is not whether to use RWP or not, but when, in which phase of the learning of new items of knowledge, and in what way.

The transfer of knowledge acquired in school into everyday or even professional life appears to be another requirement the reformists expect the school to meet. But this element seems almost impossible to pin down in research. Recent discussions concerning the need for a new understanding of transfer raise the question of the usefulness of the term of transfer itself. As we are trying to argue, its use seems to create a parallel theory of learning, which, however, rests merely on a different nomenclature and not on a real difference in phenomena distinct from the phenomena of long-term learning.

We thus understand the discourse of Czech reformists as an extraction of partial moments of the teaching/learning process. After such an extraction, these partial moments are isolated as mutually incompatible general approaches to teaching/learning – the choice is either between direct transmission of knowledge by the teacher, to name but one such purported general approach, or, as its opposite, the construction of knowledge by students. The tendency to postulate different moments as antagonisms and their labelling as universally good or bad is also apparent in constructivism-inspired reforms of teaching mathematics in the USA and elsewhere as well as in research literature.

To put undue emphasis on one element in the real yet to no extent absolute dichotomy, or, indeed to portray that element as the only desirable type of approach, inevitably reduces the efficiency of teaching/learning. Decades of research, including recent enquiries, prove that there is no panacea, no teaching/learning method which would be effective under all circumstances. By the same token, however, we believe that there is no method of which one could under all circumstances say: “If it fails to help, well, at least it will not do harm.” All methods are capable of causing damage to the process of teaching/learning if they are applied without regard to the process as a whole, in excessive measure (one-sidedly) and without repeated consideration of both the cognitive level attained by the student and of the zone of his/her proximal development.

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Exploring the Cognitive Dimension of Teaching Mathematics through Scheme-oriented Approach to Education[1]

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Abstract: Following H. Freudenthal, E. Fischbein and many other mathematics educators, we consider the main way to improve the teaching of mathematics lies in changing teacher's educational styles to develop pupils' creativity and intellectual autonomy. In our interpretation, this means using scheme-oriented education based on the theory of generic models we describe. On the basis of these ideas, a tool for the analysis of teachers' teaching style is developed. The tool and its application are described in Jirotková's article in this issue.

Keywords: qualities of teaching mathematics, teachers' pedagogical beliefs, scheme-oriented education, theory of generic models, reflection, teaching style, transmission and constructivist educational styles, goals of mathematics teaching

1 Rationale and Formulation of the Problem

This study is, in a sense, a contribution to the on-going discussion in the society about the quality of the teaching of mathematics at schools, stemming from a several year long decline of Czech pupils' results in international studies of TIMSS and PISA. Even though much criticism of drawing hasty conclusions from these studies has appeared (Štech, 2011), results of McKinsey's study (2010) clearly call for the improvement of teaching mathematics in the Czech Republic and claim that "… it is necessary to change attitudes and behaviour of people − in this case of more than 100,000 teachers − and that is a task extremely complicated for any institution." (McKinsey & Co., 2010, p. 4) After more than forty years of investigating the teaching of mathematics, we see the main deficiency to be the focus on reproductive and imitative activities in connection with a low representation of pupils' creative activities in teaching. This deficiency has been around for a long time and many documents of the Ministry of Education as well as research studies have been pointing out the necessity to enhance creativity in the teaching of mathematics for more than fifty years.

From the late fifties until the early eighties of the last century, there was an attempt in many developed countries to bring creativity to mathematics lessons by curriculum change (Adler, 1972; Hilton, 1977). This initiative under the name New Math came from the USA where "Sputnik crisis" caused a pressing need to enhance

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the professional level of engineers in the country. At the same time, A. N. Kolmogorov (1973) in the USSR and such authorities as G. Papy (1963), J. Dieudonné (1961), H. Freudenthal (1973), R. Thom (1973) in Western Europe urgently required changes in the teaching of mathematics at primary and secondary schools. The idea of changing the curriculum was the strongest. Its advocates believed that if a traditional curriculum dealing with algorithms of calculations was replaced by an approach based on set theory, the transmissive way of teaching[2] would change into a dialogic and creative one. The expected change indeed came (Kabele, 1965/1966; Kabele, Kníže, 1969; Hruša, 1968/1969). The new curriculum was very successful mainly in pupils' motivation. Previous fear of mathematics was replaced by enthusiasm of both pupils and teachers. Mathematics became one of the most popular subjects. However, the enthusiasm lasted only a few years. Gradually, the creative zeal of teachers and pupils diminished and drill came back to schools. In 1973 in the USA, M. Kline's article *Why Johnny Can't Add: The Failure of the New Mathematics* appeared and presaged the end of the set initiative.

In the former Czechoslovakia, sets were introduced to schools in 1976 and wound down in the nineties. A new set of textbooks only appeared after 1990, however, this did not lead to a marked improvement of the teaching of mathematics, rather to the contrary, as D. Greger (2011) comments on a general level.

Why did the journey of set mathematics in all the countries where it was introduced have such a dramatic course? The answer will allow us to learn an important lesson from this extensive experience. First of all, the introduction of sets was frustrating for many teachers as they had to learn new things. They could no longer instruct pupils as there were no instructions to give. They had to lead discussions, solve various problems, being creative. They became the bearers of the climate of creativity. They themselves looked for answers and they conveyed their curiosity to their pupils. And in return, pupils' unprecedented activity motivated teachers. All looked bright. But when the teachers after a couple of years mastered the subject matter, they started to economize their work and create conventions and rules as a supposed tool for the more effective educational process. Curiosity and joy from discovery left mathematics lessons. Pupils were again required to reproduce and imitate. Fear and boredom returned to mathematics − fear of those who could not work at such a required pace and boredom of those who felt the need to solve markedly more difficult problems.

Set based curricula have left schools in all the countries where it was introduced, including the Czech Republic. It was known that the return to the previous situation would not solve the original problem how to stress creativity and lessen memorisation in the teaching of mathematics. New initiatives have proceeded in several directions. Our research belongs to those following Hans Freudenthal's motto: "Mathematics is a human activity." We try to bring to life ideas which he formulated more than forty years ago: "Mathematics … is an activity of solving problems, of looking

[2] The term transmissive education is used in the sense of Askew at al. (1997).

for problems, but it is also an activity of organizing a subject matter. This can be a matter from reality which has to be organized according to mathematical patterns if problems from reality have to be solved. It can also be a mathematical matter, new or old results, of your own or others, which have to be organized according to new ideas, to be better understood, in a broader context …" (Freudenthal, 1971, p. 413–414).

Worldwide experience with New Math has shown that understanding mathematics is not given by the content but rather by the method of teaching. This had led to deep investigation of pupils' cognitive processes, see, for example, Sfard (1991), Schonfeld (1992), Dubinski, McDonald (2001), Hershkowitz, Schwarz & Dreyfus (2001). Many important research results have not found their way into practice. Thus, research in mathematics education has focused on the teacher's personality, the teacher being the main actor in the teaching process. The teacher's value system and his/her pedagogical beliefs have come to the fore, as the main parameters of the quality of teaching.

In the last twenty years, teachers' beliefs have been investigated in many studies in mathematics education. Their philosophical underpinning can be found in the work of P. Ernest (1991) and, for example, A. Thompson (1992). We focus only on those studies which explore the relationship between teachers' beliefs and teachers' practice and who are most often quoted in the literature i.e. Törner, Pehkonen, Goldin (i.e., proceedings edited by Leder et al., 2002), Speer (2008) and many others.

A teacher who feels that a traditional transmission style (see below) is ineffective and looks for a more effective one will improve his/her work if such a style is offered to him/her. However, according to our experience, there is a lack of such teachers. The teacher who is satisfied with his/her transmissive teaching must be persuaded that his/her work will be more attractive for him/her, more successful for his/her pupils and more joyful for everybody when drill is suppressed and creativity enhanced which develops a pupil's mathematical thinking. The consequence of this observation is a challenge: Look for ways to shift the teacher's beliefs so that he/she stresses creative aspects and tones down transmission aspects in his/her teaching.

A change of a person's beliefs intervenes with his/her hierarchy of values and any shift is difficult, sometimes even impossible. Our experience shows that not all teachers can be influenced and that ways to shift pedagogical beliefs can vary from teacher to teacher.

The goal of the study is to look for answers to the following three questions which concern the teaching of mathematics and an effort to shift the teacher's educational style towards a constructivist one[3]:

1. Which phenomena determine the quality of the teacher's work in teaching mathematics?
2. How to identify the teachers for whom a shift of the educational style seems to be promising (hopeful)?

[3] The term constructivist education is used in the sense of (Noddings, 1990).

3. How to find out which ways can lead to the required change for a particular teacher?

2 Cognitive Goals of the Teaching of Mathematics

In view with Freudenthal's (1973), Fischbein's (1999) and many others' ideas, we characterise effective teaching of mathematics by three cognitive goals:

- a pupil understands mathematics, his/her knowledge is not mechanical;
- a pupil is intrinsically motivated for work, he/she is not frustrated by mathematics;
- a pupil develops intellectually; by that, we mainly mean the development of the ability to: 1. communicate mathematically both orally and in writing, 2. cooperate in a group or even lead a group to solve mathematical problems, 3. analyse a mathematical problem situation, 4. effectively solve mathematical problems and 5. correct one's own mistake.

2.1 Understanding mathematics

A pupil who knows a formula, a rule or a definition or a pupil who can calculate quickly and precisely but cannot answer the question *why* his/her procedures work has mechanical knowledge only. Typical features of such a prosthetic piece of knowledge are: discontinuity from other pieces of knowledge, quick forgetting, inability to self-correct mistakes. A pupil with superficial knowledge of mathematics can be successful in standard tests but cannot solve non-standard problems and does not understand mathematics.

Thus for teachers it is important to know how a pupil comes to understand mathematics and how it is possible to diagnose this understanding. Both stem from the mechanism of a concept development process. The one which will be used here was outlined more than fifty years ago by Vít Hejný (1942). His ideas were further developed in the Bratislava seminar of didactics of mathematics and the new results were continuously tested in several classes from 1975 to 1989. Since 1992 this theory is developed by the work of the author and his collaborators at the Faculty of Education, Charles University in Prague.

The building blocks of learning with understanding are *generic models* which, in a person's mind, create a complex multi-layered dynamic structure organised into *schemes*. Both terms will be explained below.

2.2 Theory of generic models

Our model of the process of gaining knowledge is based on five stages. It starts with motivation and has as its core two mental shifts: the first leads from concrete knowledge (isolated models) to generalised knowledge (generic knowledge) and the second from generic to abstract knowledge. The permanent part of this process of gaining knowledge is that of crystallisation, which involves integrating new knowledge into the already existing mathematical structure (Hejný, 2011a).

The whole process is shown in figure 1.

abstract knowledge → crystallisation
↑ *abstraction*
generic model(s)
↑ *generalisation*
motivation → isolated models

Figure 1 Scheme of the process of gaining knowledge according to the Theory of Generic Models

Only the first three stages will be explained in detail because they are essential at elementary school level.

Motivation. We see motivation as the tension which occurs in a person's mind as a result of:

- the need to repeatedly experience joy from intellectual success which comes after solving a problem or even the discovery of a new truth, and
- the discrepancy between the existing and desired states of knowledge. The discrepancy comes from the difference between 'I do not know' and 'I need to know', or 'I cannot do that' and 'I want to be able to do that'.

Isolated models. Models of a new piece of knowledge come into mind gradually and have a long-term perspective. For instance, the concepts of place value, negative number, or straight line develop over many years at a preparatory level. For more complex knowledge, the stage of isolated models can be divided into four sub-stages as described in (Hejný, 2011a, 2011b).

This stage ends with the creation of the community of isolated models. In the future, other isolated models will come to a pupil's mind, but they will not influence the birth of the generic model.

Generic models. In figure 1, the generic model is placed over the isolated models indicating its greater universality. The generic model is created from the community of its isolated models and has two basic relationships to this community:

1. it denotes both the *core* of this *community* and the core of *relationships* between individual models, and
2. it is an *example* or *representative* of all its isolated models.

The first relationship denotes the construction of the generic model; the second denotes the way the model works. The absence of a generic model leads to mechanical knowledge. For example, the pupil who creates a generic model of the formula for the area of a triangle by solving a series of problems understands the formula. The pupil who takes the formula over by transmission has mechanical knowledge only. He/she can apply it on standard problems, however, when he/she forgets the formula there is no way for him/her to rediscover it.

2.3 Scheme

This term will be first illustrated by an everyday situation. When someone asks you about the number of doors, windows or lamps in your flat or house, you will probably not be able to give an immediate answer. However, in a little while you will answer the question with absolute certainty. You will imagine yourself walking from one room to another and counting the objects in question. Both the required pieces of information and many other data about your dwelling are embedded in your consciousness, as a part of the scheme of your flat. We use schemes to recognize not only our dwellings but also our village, our relatives, interpersonal relationships at our workplace, etc.

From the point of view of mathematics education the key question is: How did the scheme of our flat enter our consciousness? How was it created? The answer is obvious. This scheme is not the result of learning the curricular topic “The furnishing of our flat” in September, “Lamps and carpets” in October, “The Kitchen” in November etc. at school. The scheme is an outcome of our everyday experience in the given environment. In some activities our attention focuses on some part of the flat (we hang up a picture, clean windows, move furniture, tidy up, …) but all of these particulars are perceived as parts of one whole.

This implies that efficient mathematics education should be adapted to the natural process through which we learn in real life: various mathematical environments should be created and the pupil should be allowed to move in them, i.e., to solve a variety of problems in the environment. Our experience with scheme-oriented education clearly shows that a well-developed environment that allows pupils to move freely in it results in 1) substantial improvement of the pupils’ attitude to mathematics, and 2) markedly better mathematical knowledge.

The notion of scheme was elaborated in cognitive psychology by Anderson (1983) and we take it over in a way described by Gerrig (1991, pp. 244–245):

> “Theorists have coined the term schemes to refer to the memory structure that incorporate clusters of information relevant to comprehension … A primary insight to scheme theories is that we do not simply have isolated facts in memory. Information is gathered together in meaningful functional units.”

The notion of a substantial learning environment was introduced in mathematics education by Wittmann (2001) who stressed its basic property: it enables us to formulate a series of problems which help a pupil to understand deep ideas of mathematics. The concept of a deep mathematical idea is elaborated in Semadeni (2002). In our approach, three additional requirements are placed on a mathematical learning environment: connection to a pupil’s life experience, long-term nature (it is usable for pupils of different ages, at best from grade 1 to grade 12) and differentiated nature (it enables to pose problems to cater for needs of individual pupils). Wittman’s idea of substantial learning environments has found followers in the Czech Republic, too (Jirotková, 2004; Hejný, Jirotková, & Kratochvílová, 2006; Hejný & Slezáková, 2007; Hejný & Jirotková, 2004, 2007, 2009a, 2009b, 2010; Hošpesová et al., 2010; Tichá & Hošpesová, 2011; Jirotková & Marchini, 2011).

2.4 Scheme-oriented education

Scheme-oriented education is based on the construction of different schemes that interlink, combine and form a dynamic network of a pupil's mathematical knowledge and skills. For example:

- *area* starts with generic models of the area of square, rectangle, triangle, … within the environments 'tessellation', 'paper folding', 'grid paper', 'geoboard' and 'stick shapes';
- *small natural number* starts with generic models of address, status, operator of change and operator of comparison within environments 'stepping', 'money', 'pebbles', 'rhymes', 'ladder', …;
- *fraction* starts with generic models of 'one half', 'one quarter', 'divide into halves', 'equal sharing', … within environments 'pizza', 'stick' and 'chocolate'.

The traditional transmissive teaching of mathematics is based on a teacher's exposition. The teacher introduces pupils to concepts, relationships, processes and situations and leads them towards remembering definitions and formulas and imitations of procedures and algorithms. As a result of this educational style, most pupils cannot discover mathematics, their creative activity is limited and their insight into mathematics suffers from a lack of understanding of the subject matter.

For scheme-oriented education, it is necessary that a pupil is intellectually autonomous in the sense that he/she discovers new ideas or gets to them by communicating with classmates or takes them over from the classmates. It follows that a teacher's role is indispensable for scheme-oriented education and now we consider this.

3 Realisation of Cognitive Goals in the Teaching of Mathematics

A teacher is a key agent in the realisation of the above cognitive goals. It is the teacher who presents pupils with adequate problems and organises their discussion in such a way that it proceeds to the required knowledge.

3.1 Scheme-oriented vs. transmission educational style

A difference between the two educational styles can be best illustrated by the following simulated dialogue.

Story A

Two 5th grade teachers, Alena and Anezka, are speaking about their experience with the introduction of a divisibility test by 9.

> Alena: "Already in grade 3, when pupils learn to divide, they get dozens of tasks of the type *What can digit X be so that the number 4X2 is divisible by nine?* Then in grade 4, they solve a task again, for example, *Find a three digit number XYZ divisible by nine*

so that X + Y + Z = 8. Pupils find out that such a number does not exist but if $X + Y + Z = 9$, then there are a lot of such numbers. At this point, some pupils formulate the divisibility test by 9, so far only for three digit numbers. And only now in grade 5 when a half of the class discover the rule for three digit numbers, some pupils discover the same rule for four and five digit numbers. At the end, they will state a general rule and justify it with the help of a calculator for numbers with up to 10 digits."

Anezka: "Your way is interesting but far too long. First for three digit numbers, then for four digits, then five digits – it requires a lot of time. Moreover, I doubt that weaker pupils understand what it is about. They do not have time to practise the rule sufficiently. Of course, for two or three best pupils, it is inspiring, but most pupils, at least in my class, would require a clear rule which they acquire easily. By the way, you speak about dozens of problems, do you have a collection of such problems?"

When asked, Alena is able to show dozens of tasks which pupils solved in grade 3 and 4 and for some, she also shows erroneous pupils' hypotheses and examples from the follow up discussions in the class.

Anezka's last sentence shows that mathematics educators have an important task: to create sets of sufficiently varied (in terms of content and difficulty) problems to topics of school mathematics which would enable pupils to discover generic models via isolated models. This can easily be done by didactic mathematical environments (Hejný, 2011b).

3.2 Teacher's role

The teacher's work within the scheme-oriented education is guided by the following principles.

The teacher should

1. create optimal climate for learning: no pupil is frustrated, no pupil is bored, the teacher shares the pupils' successes, the teacher encourages pupils who might give up mathematics, the teacher builds the pupils' self esteem;
2. leave pupils space for their considerations: does not impose his procedures in their minds even when the pupils' ones are clumsy, does not orient pupils towards a quick solution by overly leading questions, does not interrupt pupils' thinking processes, when a pupil asks a direct mathematical question, the teacher values the question "It is an interesting question" and asks the class to look for answers;
3. lead pupils towards discussions: when hosting the discussion he/she makes space for erroneous ideas and weaker pupils. For example, when two pupils discover two different correct algorithms for written addition, the teacher lets each one choose which one he/she will use;
4. not point out mistakes to pupils: let pupils discover the mistakes, or provides them with a suitable task where they can see them; the mistake is understood as a way towards a deeper understanding of the investigated situation, he/she teaches pupils to analyse mistakes, mainly by analysing their own mistakes;
5. provide pupils with adequate tasks: each pupil, both of a high and low ability, solves the task which corresponds to his/her abilities and thus can experience

joy from success; problems assigned which do not allow for differentiation are frustrating for weaker pupils and boring for high ability pupils. On the other hand, the tasks which allow for both "quick" and "slow" solving strategies are suitable (see story C);

6. lead pupils (by his/her own approach to mathematics) to the need to understand not only mathematics as such but also the way it might be understood by their classmates: when the problem is solved by various solving strategies, the pupils broaden not only their horizons in mathematics, but also their understanding of other people's thinking processes and opinions; in this sense, mathematics contributes to critical thinking and cultivates pupils' democratic awareness.

3.3 Illustration

The illustration describes the first 9 minutes of a video recorded in grade 3 elementary class in February 2012. There were 7 boys and 14 girls in the class. The teacher J. M. has been teaching the pupils since September 2011. The video recording was made by a student A. Sukniak who happened to observe the lesson.

Story B

The teacher set the task: *Mother bought 5 lollypops, 3 CZK each, and chocolate for 12 CZK. How much the purchase cost?* Soon, the pupils lifted scratch writing tablets with their solutions. The teacher observed the tablets and after one minute, she wrote three results on the blackboard without the evaluation of their correctness: 21, 22, 27. Then she asked Lucka to explain the result of 21. When explaining, the girl herself found out that she made a mistake in 3×5 equals 9 and corrected the result into 27. Then the teacher without reacting to the approving voices from the class asked Lukas to explain his result of 22. The boy immediately said that it was not correct and corrected his result. Three minutes passed from setting the task.

The teacher asked the pupils to write their strategy on the blackboard. Three recordings appeared in two minutes: the first $5 \cdot 3 + 12 = 27$, the second $5 \cdot 3 = 15$, $15 + 12 = 27$, Lukas's, $5 + 5 + 5 = 15 + 12 = 27$. There were only approving commentaries to the first two: "Yes, I have the same." "It should be written one below each other." Some protests could be heard for the last record. The teacher asked Jolana: "What do you not like about it?" Jolana said that $5 + 5 + 5 = 15 + 12$ was not correct. The teacher asked Lukas if he understood Jolana. Lukas stated that not at all. Then Kristyna made the second attempt to explain the mistake, in a clear and persuasive way: "Five plus five plus five is fifteen; it is true up to here; when you add twelve, it is no longer true." However, Lukas did not understand this explanation either. The teacher asked the class who would explain it to Lukas. Four girls put their hands up and the teacher called out Misa: "In fact, he had one problem. He had it in one problem, but when Lucka puts it into two problems, then it is no longer the one problem and it does not hold that twelve plus fifteen equals fifteen." Lukas: "I do not understand this one at all." Misa took Lukas to the blackboard, covered number 27 with her hand and read backwards what Lukas had written: “Twelve plus fifteen is five plus five plus five, it is true?” After some hesitation Lukas said: “No.” Misa: “So why did you write it?” Lukas laughed and went to his desk. Nine minutes passed. The end of the story.

The story illustrates five of the six principles above.

1. No pupil was unnecessarily frustrated, the teacher was not nervous and did not press Lucka who spoke quite slowly. As far as we can say from the observation and the video recording, no one was bored, not even quick Misa. Misa, as transpired later, was solving a meta-cognitive problem of Lucka’s and Lukas’s mistake.
2. The teacher let each pupil explain his/her strategy. She gave them space for their considerations in which she did not intervene. She let the pupils look for the explanation for Lukas and did not enter the discussion even after the first two unsuccessful attempts. It is obvious that after common four month work, a didactic contract (Brousseau 1997; Sarrazy, Novotná 2005) between her and the class had set up in which the pupil addressed by the teacher did not feel fear but an opportunity to express himself/herself.
3. The teacher first gave precedence to two erroneous results and only after the explanation of mistakes she asked pupils to write strategies on the board. This encouraged a 6 minute discussion which the teacher only moderated. When observing their classmates, the pupils acquired experience with other solving strategies.
4. The teacher did not point to any of the three mistakes which appeared. The pupils identified them and removed them. The boy with the result of 21 and Lukas realised the cause of the mistake, too. None in the class found out the inaccuracy in Lukas’s record. The price of five lollypops 3 CZK each should have been written 3 + 3 + 3 + 3 +3 and not 5 + 5 + 5. The teacher did not point to it because she wanted to keep the dynamics of the lesson.
5. The fifth principle could not be seen in story B because the task used was an introductory one, presented to the whole class.
6. From frequent comments from the class to the records on the board, it is clear that the pupils followed their classmates’ work with more attention than is customary in grade 3. The discussion among pupils bore evidence of the fact that their attitude to mathematics is creative and not consumerist. An excellent example of only four month impact of the teacher was Misa’s precise analysis of Lukas’s mistake and effective didactic mastering of the situation. First, Misa formulated the cause of Lukas’s incomprehension of his mistake in a not very comprehensible manner, rather to herself and the teacher, and then she demonstrated it in a short but effective dialogue with Lukas. The core of Lukas’s conviction that his strategy was correct lies in a procedural understanding of the calculation (described in literature many times). He did not understand an argument that he had to write it in a conceptual way. He realised that the conceptual record was correct, but he did not know what was wrong about his record which exactly copied the course of ideas. By reading Lukas’s procedure backwards, Misa conceptualised the solution and Lukas understood his mistake.

The fifth principle is illustrated by subsequent story C which comes from our archive.

Story C

Grade 3 pupils were solving the task which was accompanied by figure 2.

Four sticks are needed for a square. How many sticks do we need to create windows of two squares? How many sticks to make a window of three squares? Of four squares? Of five squares?

Figure 2 How many sticks?

Each pupil had enough sticks available. A similar problem was solved by the pupils a month ago when the windows were triangular.

In the first 6–7 minutes, Verka created the first two figures on the desk, she redrew them to her exercise books and wrote number 4 and 7 to them. Gusta did the same thing. Richard had already made the third shape, was redrawing it to his exercise book and whispered “ten, ten, …” not to forget that he would need 10 sticks. Most pupils proceeded further in their work. Hanka had already solved the task. She proceeded in such a way that she added another square to the created windows of two squares and counted the number of sticks. Next she added the fourth square and counted the number of sticks again. At the end, she proceeded similarly for the fifth square. She only wrote numbers 4, 7, 10, 13 and 16 in her exercise book and ran to show it to the teacher. Some other pupils went to the teacher, too. He asked them to mutually compare their answers. Those who had finished were encouraged to count sticks for longer windows. Not all the pupils had the same results, so some corrected their work. The pupils who were sure of their results were counting other windows.

Then suddenly, Tomas cried out: “It is always plus three.” He discovered that for the sequence of found numbers it holds: the subsequent number is the previous one plus three. More than a minute before that, Imrich whispered the rule to the teacher. The teacher did not comment on its correctness and asked him to find out how many sticks he would need for windows of ten, twenty, thirty, … a hundred squares. (The teacher knew that Tomas safely counted to 1000.)

The teacher addressed the class and asked them if they understood what Tomas was saying. Karla and Danka said that they found it, too, but Veronika said that she could not understand. Tomas wrote a sequence of results 4, 7, 10, 13, 16 on the blackboard and showed that his rule was valid. Several pupils started to check if we needed 19 sticks for a window of six squares. Radka confirmed that Tomas was right, that she “had already counted it”. Verka and Richard did not participate in the discussion, went on with their work and were just finishing the last window. After a while, Imrich said to the teacher that for the tenth window, 31 sticks are needed, for the twentieth one 61 sticks, for the thirtieth one 91 sticks and for the hundredth window 301 sticks. Let us add that the next day Imrich came with the rule: The number of sticks is three times of the number of squares in the window plus one. After a month, when most of the pupils knew Imrich's rule, Hanka came with the idea connected to the earlier used triangular windows. She found out that the number of sticks was twice the number of triangles plus one. The end of the story.

Story C shows that a suitably chosen task can be adequate for both mathematically weaker pupils such as Verka and Richard and for the mathematically strongest pupil Imrich and for all the others. Verka and Richard made five isolated models and were able to continue with looking for others independently. After solving the given problem, most of the pupils were able to continue with its extension. Tomas discovered a procedural generic model in the sequence of sticks. Some other pupils made the same discovery and others accepted it immediately because they were ready for it by their own activity. Imrich, who discovered this generic model as the first pupil, was assigned a task from the teacher which led him to the conceptual generic model. From this point of view, the task was chosen appropriately as numbers 31, 61 and mainly number 301 meaningfully suggested how the number of sticks relates to the number of squares. It is quite natural that different pupils reached different levels in understanding the relationship between the number of squares and the number of sticks. Only after some other pupils discovered the conceptual generic model in a few days, did the teacher ask one pupil to show the rule to the whole class. It can happen that in such a case even the pupil who only took over the discovery makes his/her own discovery at this point. We could see that in Hanka's case.

4 Conclusions

We have shown that the teaching style aimed at the pupil's intellectual development in mathematics can be realised by scheme-oriented teaching. We have justified the key role of generic models for building schemes. We have illustrated the educational strategy as a process which leads to the birth of a generic model through isolated models. We assert that the absence of a generic model leads to mechanical knowledge and the development of a pupil's creativity is determined by the process of looking for generic models. In more than ten years of research we have been oriented towards both pupils and their teachers. In this study, we have looked deeply into the diagnosis of the teacher's educational style. However, much more is needed, namely a tool which can be used for the description, analysis and comparison of teachers' teaching styles. Such a tool together with its development and application is presented in Jirotková (2012) in this issue.

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Contribution of Geometry to the Goals of Education in Mathematics[1]

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Abstract: Through visualisation, geometry can mediate understanding of some demanding arithmetic and algebraic concepts, relationships, processes and situations for pupils. This thesis is explained by the method of genetic parallel and of a didactic analysis of two educationally interesting problem situations. Theoretical considerations are illustrated by several real experiences. Suggestions for the application of theoretical results are given in conclusion.

Keywords: method of genetic parallel, language of arithmetic and algebra, *shaped psephophory*, the language of geometry, visualisation, gnomon, discovery, proof, process, concept

1 Introduction and Methodology

Geometry appears at two levels in school mathematics. At the first level, plane and space shapes, relationships, constructions, proofs, etc. are introduced to pupils. At the second level, geometry provides support for arithmetic and algebra. Pupils can be strongly dependent on the visualisation of arithmetic and algebra. For example, some pupils are able to understand the additive structure of integers already in the second grade of the primary school with the help of a number line, but without it, they cannot carry out additive operations with negative numbers even in the eighth grade. The goal of the study is to provide examples of how geometry can help in understanding arithmetic or algebraic concepts, processes, relationships and arguments. The geometric support is decisive for some pupils, not only from the point of view of understanding the subject matter but also from the point of view of their approach to learning. A superficial approach which enables pupils to "meet the requirements of knowledge reproduction" changes due to visualisation into a deep approach which enables them to "really understand the subject matter" (Mareš, 1998, p. 39).

Two methods are used in this study. The first is the method of genetic parallel which postulates that relations found in the phylogeny are inspirations for revealing relations in the ontogeny. An inspiration for the use of the second level of geometry in the teaching of mathematics can be found in the sixth century BC when the

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language of shaped psephophory shifted mathematical thinking from the area of arithmetic to geometry. We also use the results gained by the study of mathematical languages in history by L. Kvasz (2008).

The second research method consists of the didactic analysis of problem situations. For each such situation, arithmetic, algebraic and geometric approaches are shown and didactic phenomena analysed. Short stories which illustrate the analyses come from our archives.

Finally, some suggestions for the use of theoretical results in practice are given.

2 Method of Genetic Parallel

The following quotation aptly characterises the idea of genetic parallel:

"The growth of the tree of mathematical knowledge in the mind of one person (ontogeny of mathematics) will only be successful if it replicates to a certain extent the history of the development of mathematics" (Erdnijev, 1978, p. 197).[2] This idea is expressed more precisely by Freudenthal (1991, p. 48): "Children should repeat the learning process of mankind, not as it factually took place but rather as it would have done if people in the past had known a bit more of what we know now." (Freudenthal, 1991, p. 48). Jankvist (2009) illuminates the method of genetic parallel in a concise way.

Mathematical knowledge of pre-Greek civilisations from the Nile and Hindu basins and Mesopotamia answered the question *How?*. How can we calculate? How can we find? How can we construct? … It was mainly the knowledge of calculation, in the present language, arithmetic knowledge. The Greeks were the first to ask for the basis (úsia) and cause (*aiton*, *aitia*) of things and phenomena. They understood that the knowledge of causes is more important than the knowledge of instructions:

> […] we suppose artists to be wiser than men of experience (which implies that Wisdom depends in all cases rather on knowledge); and this because the former know the cause, but the latter do not. For men of experience know that the thing is so, but do not know why, while the others know the 'why' and the cause. (Aristotle, The Metaphysics I, p. 1)

Milesian philosophers in the sixth century BC explored the basis, the substance of the world. They found it in water, air or indefinite *apeiron*. Pythagoras claimed number to be the essence of the world. "Pythagoras, …, said that 'all things are numbers'. This statement, interpreted in a modern way, is logically nonsense, but what he meant was not exactly nonsense." (Russell, 1965, p. 54). Pythagoras believed that all phenomena in the world such as joy, truth, justice, courage, male principle, female principle, etc. had their own representations in the world of numbers and thus the relations of the world were depicted in the relationships among numbers. At present, we would say that the unclear and variable scheme of things is grasped

2 Рост древа математических знаний в голове отделного человека (онтогенез математики) будет успешным тогда, когда он повторяет в известной мере историю становления этой науки (филогенез математики).

by a strict and immaculate scheme of numbers. The knowledge of eternal essences, scientific knowledge (*epistémé*), is more important than practical knowledge of counting (*phronesis*).[3]

In the Pythagorean school, mathematics as a scientific discipline was born, as a discipline looking for exact definitions of concepts, general regularities and their proofs. An important by-product, possibly the key one, of this birth was the change of language. The language of pebbles (*pséfoi*), which was used for counting in the whole Mediterranean in the sixth century BC, was changed into the language of shapes (*shaped psephophory*; in the contemporary terminology figurative numbers). The number previously represented by a pile of pebbles was represented by a shape (made by spreading the pebbles into this shape). In this way, figurate numbers originated (Figure 1).

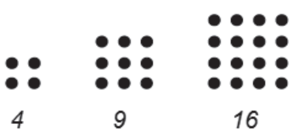

Figure 1a Square numbers

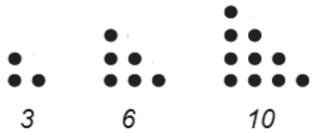

Figure 1b Triangular numbers

In this language, the square represents an infinite sequence of numbers; the same applies for the shape of a triangle. On the other hand, all even numbers can be described by a single shape, a rectangle of the width 2 and of any length; in brief *2-rectangle* (Figure 2a). Each odd number can be described by a *2-rectangle with an appendix* (Figure 2b).

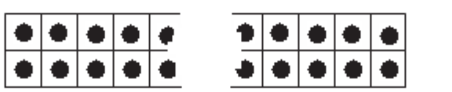

Figure 2a An even number of pebbles arranged as a 2-rectangle

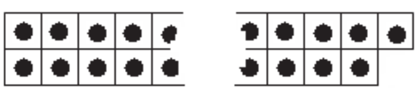

Figure 2b An odd number of pebbles arranged as a 2-rectangle with an appendix

The new language made it possible to formulate and prove general statements which went beyond the horizon of the then illuminated world of numbers. For example, the statement

The sum of two odd numbers is an even number. (*)

does not only hold for small numbers which we can imagine but also for numbers inconceivably big, such as the number of grains of sand in the desert. If we connect two odd numbers (i.e., two 2-rectangles with appendices), the two "appended" pebbles make a pair and the result is a 2-rectangle, that is an even number. The connection is in Figure 3.

3 F. Korthagen (2011, p. 36–44) elaborates this typology in more detail.

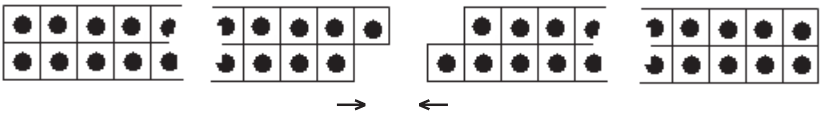

Figure 3 The connection of two odd numbers results is a 2-rectangle

The presented proof of the statement (*) holds for all numbers. That is guaranteed by the fact that the length of the 2-rectangle with an appendix (both right and left ones) can be of any measure.

It is interesting to note that Pythagoras could have stated the same idea without the use of geometry. He could have defined an even number as any number of pairs of pebbles and an odd number as any number of pairs of pebbles plus one pebble. Why did Pythagoras not do so? Why did he introduce the language of geometry? Most probably it was because the arithmetic pairing was understood in a procedural way as manipulation, which appears to be unrealisable for big numbers. The geometric shape is understood in a conceptual way as a ready-made object which exists regardless whether we can or cannot imagine it; an object which we know through its properties, not through sensory experience; an abstract object. And thus the birth of mathematics is connected to the change of language: the number, previously understood in a procedural way as a pile of pebbles, is now understood as a concept, as a geometric shape.

The change of language is a key constituent of the historical development of mathematics. L. Kvasz (2008, p. 16) presents six points of view from which he studies the development of the languages of mathematics:

1. logical power – how complex formulas can be proven in the language;
2. expressive power – what new things can the language express, which were inexpressible in the previous stages;
3. explanatory power – how the language can explain the failures which occurred in the previous stages;
4. integrative power – what sort of unity and order the language enables us to conceive there, where we perceived just unrelated particular cases in the previous stages;
5. logical boundaries – marked by occurrence of unexpected paradoxical expressions;
6. expressive boundaries – marked by failures of the language to describe some complex situation.

If we apply the first five points on the birth of shaped psephophory as a new language, we can see that there is a series of cognitive and meta-cognitive shifts:

1. from items to generality and from the concrete to the abstract;
2. from the work with numbers before the horizon (small numbers) to the work with any number;
3. from the practical relationships to the theoretical ones (from *phronesis* to *epistémé*);

4. from a set of instructions of calculation, to a systematically shaped psephophory (in other words, from a process to a concept);
5. from sensory evidence to general proofs.

3 Didactic Analyses of Problem Situations

Let us now consider the second of the research methods used, the analysis of tasks which allow for both arithmetic and geometric solutions. We will show how predominantly procedural arithmetic or algebraic solutions become more understandable by the use of geometric concepts. Two illustrations will be given.

3.1 Task 1: *Prove statement (*)*.

In the arithmetic language, the parity of the number can be found through the record of the number in a decimal system: number n is even *iff*[4] its last digit is 0, 2, 4, 6, or 8; number n is odd *iff* its last digit is 1, 3, 5, 7, or 9. Let us add, that the positional decimal system was discovered more than 1500 years after Pythagoras. Nevertheless, our pupils know this powerful language and already in the third grade most of them also know that the parity of the number is given by its last digit.

An arithmetic proof of the statement (*) lies in checking the following fact: the sum of any two numbers from 1, 3, 5, 7, 9 equals the number which ends with the digit 0, 2, 4, 6, or 8. This process cannot be seen as one whole, it is necessary to keep a step by step record of it. The proof of the statement (*) requires checking 15 simple sums.

It is a mathematically correct proof which can be independently discovered by pupils from grade 4.

In the algebraic language, the parity of the number is most often given by the following characterisation: the number is even *iff* it can be written in the form of $2k$, where k is a natural number; the number is odd *iff* it can be written as $2k + 1$, where k is a natural number.

The algebraic proof of the statement (*) lies in the manipulation with the expression $(2k + 1) + (2m + 1)$ into the form of $2 \times (k + m + 1)$, that is, into the identity

$$(2k + 1) + (2m + 1) = 2 \times (k + m + 1), \qquad (**)$$

where k, m are natural numbers and thus $k + m$ is also a natural number.

The identity (**) is a concept and the proof is a correct one. We know that the proof is not understandable for many pupils of grade 9 but we also know of cases when the proof was discovered by a pupil from grade 6.

In the geometric language of shaped psephophory, the parity of a number is given by fig. 2a and 2b and the proof of the statement (*) by fig. 3. The proof is correct and pupils from grade 4 are able to discover and understand it.

[4] I.e. if and only if.

The didactic analysis of the above three proofs of the statement (*) looks into the difficulty of grasping the concepts of even and odd and into the thinking processes present in the proofs.

The arithmetic grasping of the concept of even and odd is based on a two-step process: (1) a pupil realises that for a number to be even or odd, the last digit is the key one, (2) he/she finds out if this number belongs to the set {0, 2, 4, 6, 8}, or to the set {1, 3, 5, 7, 9}. This characterisation of evenness/oddness does not have to be evident for a pupil from grade 2.

Story 1

Five pupils from grade 2 were to find an odd number whose last digit was 4. One boy started to laugh and the others immediately reacted that it was not possible. After a while, Adela said that it would have to be very big.

The story shows that the knowledge "the parity of the number is given by its last digit" is not evident for all pupils in grade 2.

The arithmetic proof is a lengthy one. It is redundant for some pupils because the situation is clear to them. However, some pupils feel the need to verify all 15 sums.

Story 2

Adela from story 1 created Table 1. She was looking at it with delight for some time and then she said "yes, now it is clear". The next day, she brought two more tables to the teacher. The first one was for the proof of the statement that the sum of two even numbers is an even number and the second for the proof of the statement that the sum of an odd and even number is an odd number. What led Adela to create the table? She felt that the series of individual calculations did not bring an insight into the proof and found the right way to acquire it. A long process of the creation of the table led to the table as a concept in her mind. Thus the table became the main bearer of the proof of the statement (*) for her. This proof, as well as the following algebraic one are axiomatic proofs from the point of view of Housman and Porter's classification (2003). The marked difference between them is described within Kvasz' theory (2008) here by the language used.

Table 1 Addition of odd numbers

+	**1**	**3**	**5**	**7**	**9**
1	2	4	6	8	10
3	4	6	8	10	12
5	6	8	10	12	14
7	8	10	12	14	16
9	10	12	14	16	18

The algebraic grasping of the concept of even and odd is based on understanding the notation of $2k$ and $2k + 1$, where k is a natural number. This characterisation of parity is not understandable for many pupils; often, they simply remember it as is illustrated by the following story.

Story 3

In grade 7, Bara discovered the algebraic proof (**). When asked by the teacher, she showed the discovery to the class. In the subsequent discussion in the class, the following statements could be heard (among others):

Cyril: What is odd then? It is $2k + 1$, or $2m + 1$?
Dana: But I found out that even (she points to the record $2 \cdot 3.5 - 1 = 7 - 1 = 6$).
Ema: But last time, an odd number was $2k - 1$.
Philip: And can I write it as $1 + 2k$, too?

The algebraic proof is brief and for a pupil who understands the identity (**) as a concept, it is clear. If the pupil does not understand the meaning of records $2k$ and $2k + 1$, he/she cannot understand the proof either. These are all four pupils from story 3 and many others from grade 7. However, even the pupils who have a good understanding of odd and even numbers in the language of letters are often not able to grasp the proof as a whole. It is illustrated by a story of one of our colleagues AŠ.

Story 4

AŠ speaks about her experience from grade 7: *I know that it is difficult, therefore I took great care that all knew how to record an even number and how to record an odd number. Then I let Gita, who is the best mathematician from the class, to make the sum* (**). *The girl did it marvellously. I asked the class if they understood. All nodded that they did. So I asked them to come and show in a similar way that the sum of three odd numbers is odd. Only two pupils put their hands up and shyly one more. I know about them that they can do it but beside them, none. It is simply too difficult for seven graders.*

Why is the algebraic proof so difficult? There are two causes. The first and key one lies in the fact that the teaching of algebra concentrates mostly on the manipulation with algebraic expressions, that is, the manipulation with the letters. Little attention is paid to the clarification of the meaning of this language.

Story 5

In grade 8, Hanka is solving a task at the blackboard:

How long does it take a cyclist who goes at the speed of $v = 16$ km/h to cover the distance $s = 10$ km?

After a while, the girl says: “I have forgotten the formula.” The teacher writes $v = s/t$ on the board. The girl says: “I know it and I also know that $s = v \times t$, but I have forgotten the third one.” The teacher is lost for a moment what to do and then she asks the class: “Could anyone help Hanka but without actually saying the formula?" Ivan says: "Hanka, write the equation $4 = 3/x$ and solve it." Without any problem, Hanka finds $4x = 3$ and $x = ¾$. Ivan continues: "Now you write $a = b/x$ and solve it." Again Hanka finds $x = b/a$. Ivan: "Excellent. Now, write $v = s/x$ and when you solve it, use letter t instead of x." Hanka writes the equation, compares it to what she has been doing earlier, laughs and without solving anything she says "Yes, I have remembered," and writes $t = s/v$.

In the context of equations, the girl had no problems with the algebraic manipulation but she had no clue that the same manipulation could also be made in a different context. The story shows an inappropriate approach to the introduction of letters to pupils. The manipulation with letters has no support in semantic ideas. In the case of proof (**), it is possible to find such a support just in the psephophory geometric proof as Figure 4 depicts.

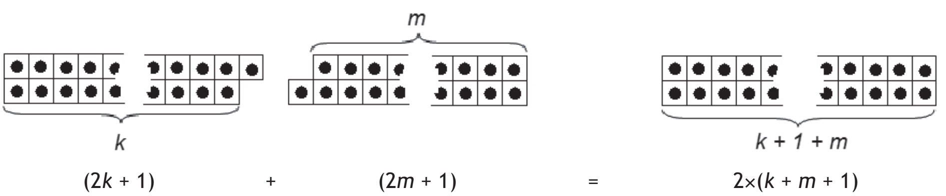

Figure 4 Geometric proof

The above didactic solution of the problem how to mediate the understanding for the identity (**) for pupils is similar to the didactic solution of the problem how to teach pupils to solve word problems. For word problems, dramatization helps, that is, mathematical phenomena are grasped in a semantic way, they are supported by a pupil's real life experience. We have used visualisation for the identity (**), that is, we supported algebraic objects by a pupil's geometric experience. In both cases, we can speak about the semantic way of grasping the situation if we suppose that geometric objects we work with are for pupils evident in the same way as life experience at word problems. The pupil who does not have this evident knowledge of geometric shapes such as rectangle and shapes on a square grid cannot use the way via visualisation of the relation (**) successfully. He/she is in the same situation as a pupil who is solving a word problem about dairy cows but does not know the word dairy cow.

We have finished our considerations about the first task. The second task often appears in secondary textbooks.

3.2 Task 2: Find the sum of the first n odd numbers: $s_n = 1 + 3 + 5 + \ldots + 2n - 1$.

The arithmetic solution is based on the experiment, observation and generalisation. The pupil finds out that $s_1 = 1$, $s_2 = 4$, $s_3 = 9$, $s_4 = 16$, … and notices that the given numbers are squares and that in all the cases, the result is $s_n = n^2$. Thus

$$s_n = 1 + 3 + 5 + \ldots + 2n - 1 = n^2. \qquad (***)$$

The pupil considers this generalisation of four observed cases to be the solution. He/she might check it by two or three more calculations. Even though this solution is not proper, the result is correct and the pupils believe it. This proof is a typical example of inductive reasoning which is characterised by Housman and Porter (2003) as: "A student with an inductive proof scheme considers one or more examples to be convincing evidence of the truth of the general case." (p. 40)

The algebraic solution is based on the manipulation of expressions. If the pupil adds the first and last elements, i.e., $1 + (2n - 1)$, the second and the last but one elements, i.e., $3 + (2n - 3)$, etc., he/she finds out that each from the sums is $2n$ and that if n is an even number, there are $n/2$ of these sums. Thus for n even, it is $s_n = 2n \times n/2 = n^2$. Then the pupil finds out that for n odd, the result is the same.

In both cases, arithmetic and algebraic ones, the pupil gets to the result of $s_n = n^2$. However, the mathematics teacher is sometimes not satisfied with this result and asks for the proof. Few pupils understand what the teacher is looking for. Such a pupil shows the proof by mathematical induction[5], which the teacher is satisfied with but most pupils, in fact, do not know what it is about.

The geometric solution is based on the visualisation of the expression. An important role is played by the shape which the Greeks called *gnomon* and which was commonly used by Euclid. We will explain it. If we cut out a square from another square so that both squares have a common vertex, then the remaining shape was called gnomon by the Greeks. If the side of a bigger square is by k (pebbles) bigger than the side of the smaller square, we say that this gnomon is of width k, in brief k-gnomon. Figure 5a shows 1-gnomon and Figure 5b 2-gnomon. We can see that each 1-gnomon is an odd number and each odd number is 1-gnomon. Thus, the sum of several odd numbers is made of several gnomons.

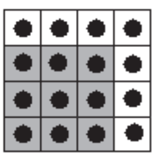

Figure 5a 1-gnomon

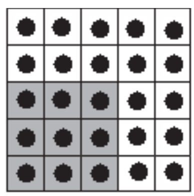

Figure 5b 2-gnomon

Figure 6 shows the solution to the task by shaped psephophory. The figure is made like this: first, one pebble is placed, with 3 pebbles around it in the shape of 1-gnomon, there are 5 pebbles in the shape of 1-gnomon around this square of 4 pebbles, etc. It is clear that by gradual adding of 1-gnomon, the resulting square "grows" but remains the square.

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5 Mathematical induction is a method of mathematical proof typically used to establish that a given statement is true for all natural numbers (positive integers). If $V(n)$ is a statement which is true for all natural n, then we can prove that $V(n)$ holds for all natural numbers by proving the statement $V(1)$ and the implication $V(k) \Rightarrow V(k+1)$ for all natural k.

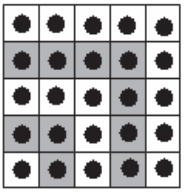

Figure 6 Solution by adding of 1-gnomon

Even against this geometric solution we can protest by saying that the statement has not been proved. The proof would be made by mathematical induction again, this time in the context of shaped psephophory.

The proof will be described in a hypothetical dialogue between two pupils.

Story 6

Jana, which discovered the relationship $s_n = n^2$ in class in an arithmetic way, boasted about her success to her brother. The brother showed her how to derive the relationship from fig. 6. Jana liked it and asked the teacher to be allowed to show it in class. The class liked the figure, too, but Karel protested: "But what if the next gnomon does not fit the preceding square for big numbers? What if it is a bit shorter or longer?" The teacher suggested that the pupils thought about this objection. Jana: "Let us presume that until number k it will work. So the sum of the first k odd numbers is a square of side k. What 1-gnomon follows? The last gnomon had $(2k - 1)$ pebbles and that is why the following has $(2k + 1)$ pebbles. And that is just what we need to get a square of side $k + 1$ from the square of side k."

Jana's proof is based on mathematical induction. As it is an illustrative one, pupils will understand it better than the algebraic one.

4 Conclusions

The study analyses in a didactic way two tasks in which geometry markedly helps understanding an arithmetic situation. A deeper consideration of the cognitive structure of this didactic problem shows that geometry helps to transform arithmetic or algebraic thinking of a procedural nature to the conceptual level. In task 1, the proof was divided into many partial steps in the arithmetic language and too sophisticated in the algebraic language to be grasped by most of the pupils. In the geometric language, the proof was short and clear. Task 2 concentrated on finding the formula $s_n = n^2$. The arithmetic language leads a perceptive pupil to the solution via isolated and generic models (Hejný, 2012). However, it does not provide persuasive argumentation. Again, the algebraic language requires sophisticated considerations. The geometric language shows in a single picture both the process of the growth of the resulting square and the argumentation described in story 6. In this story, we can also see the geometric contribution on a meta-cognitive level: the proof by mathematical induction, which is in its arithmetic or algebraic realisations non-understandable to most of the pupils, is much easier to grasp in a geometric context.

We should add that the visual support of arithmetic and algebra can be widened by dramatization for kinaesthetic support. For example, task 3 can be solved by pupils from grade 4 by walking on a staircase with numbered steps (i.e., on a number line).

Similar didactic analyses as the above for tasks 1 and 2 can be made for many different tasks such as:

Task 3. Solve the equation $|1 - |x + 1|| = 2$.

Task 4. Find the sum $s_n = 1 + q + q^2 + \cdots + q^n$, where n is a natural number and $q > 0$ is a real number.

Task 5. Find the sum of an infinite series $s_n = 1 + q + q^2 + \cdots + q^n$, where $0 < q < 1$.

Task 6. Prove that $\sin\alpha + \sin\beta = \sin\alpha \cos\beta + \cos\alpha \sin\beta$, for $0 < \alpha, \beta$ and $\alpha + \beta < \pi/2$.

Our experience as well as the experience of many cooperating teachers show that visualisation is a crucial scaffold for securing understanding of concepts, relationships, situations and processes in mathematics for some pupils. The problem of the use of geometry for the understanding of phenomena in arithmetic, algebra, but also combinatorics, probability or logic is elaborated in many publications. From one of them, which has become a classic, we choose one fitting verse to conclude with:

Geometry is to open up my mind
so I may see what has always been behind.
Henderson (2001, p. *ii*)

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A Tool for Diagnosing Teachers' Educational Styles in Mathematics: Development, Description and Illustration[1]

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Abstract: In his article, M. Hejný (this issue) provided the theoretical background for the development of a diagnostic tool which can be used for the characterisation of a mathematics teacher's educational style. The goal of this study is to describe the tool and show how it can be applied. The tool consists of 20 parameters divided into four areas: (A) beliefs, (B) experience, (C) personality and (D) abilities / competences. The tool has been applied to two short case studies and one longer one. This tool can also inspire a teacher for the improvement of his/her teaching style.

Keywords: diagnostic tool, quality of teaching mathematics, a teacher's pedagogical beliefs, a teacher's experiences, a teacher's personality, a teacher's abilities, reflection, transmission and constructivist educational styles, goals of mathematics teaching

1 Introduction

This study originated within research led by M. Hejný and stems from his study in this issue (2012). On the basis of his ideas, we have developed a diagnostic tool which can be used to characterize a mathematics teacher's educational style. The tool focuses on the measure to which the teacher develops pupils' creativity in his/her teaching. It enables the teacher to uncover his/her deficiencies in the area of pupils' creativity and points to the possibility of improving his/her work. The researcher can use the tool to analyse teachers' educational styles in detail.

The tool consists of 20 parameters divided into four areas: (A) *Beliefs*, (B) *Experiences*, (C) *Personality*, (D) *Abilities/competences*. The goal of this study is to introduce the methodology of research leading towards the tool, describe the tool in detail and illustrate its application with examples.

When investigating the teacher's educational style, we focused on three target groups: 1. qualified in-service teachers, 2. in-service teachers studying to finish their qualifications, 3. pre-service teachers studying at the author's faculty to become elementary school teachers.

In the last ten years, we have acquired a rich and varied set of research data: video-recordings of lessons, audio-recordings of seminars with teachers, seminar works about the teachers' experience, master theses, written reflections from

1 The study was supported by research project No. MSM 0021620862 Teaching profession in the environment of changing education requirements.

teachers and students on the presented educational situations, written reflections and self-reflections of lessons, pupils' written work, some of them with the teacher's comments, records from observations, etc. Our colleagues also supplied our database with interesting and valuable data.

The rich research material has been analysed from the point of view of pupils' thinking processes and class discussions (Hejný, 2004a, 2011; Jirotková, 2011; Jirotková, Kratochvílová, 2004) and from the point of view of a teacher's educational style (Hejný, 2004b, 2004c, 2004d, 2007, 2008; Kratochvílová, 2004). Various techniques of qualitative analyses have been used (comparative, genetic, classification, etc.), mainly based on techniques of grounded theory (Strauss, Corbinová, 1999).

The research into the teacher's educational style began by the study Hejný (2006) which was inspired by diversity idea of Gray and Tall (Gray at al. 1999): the more creative the pupil, the more varied his/her reactions to challenges. Our development of this idea suggests that: the more creative a teacher's educational style, the more varied the solving strategies of his/her pupils.

2 Description of the Tool

Due to the nature of the investigated phenomena, the diagnosis tool which is described here cannot be considered final. The research is on-going and both the set of parameters and their classification into individual areas can be modified with the new data and their analyses. We often faced a problem how to classify a certain phenomenon. For example, the phenomenon of self-assessment is included in the area of Personality, but it can also be put into the area of Beliefs.

Area A – teacher's beliefs – seems to be the key one. It is, however, the result of the teacher's life experience, that is part B. A teacher's personality, area C, is connected to area A in such a way that it is not often possible to include an idea to only one of them. The last area, D, covers the teacher's abilities / competences and is closely connected to the educational processes. This area can be seen as pedagogical content knowledge and content knowledge in the sense of Shulman (1987; see also Janík, 2009). We argue that it is not necessary to define precisely the terms such as belief, life experience, personality, value polarities, etc. as we do not suppose that misunderstanding could arise. As for the key term belief, a concise summary of key research in this area is given in Žalská (2012).

At present, the proposed set of parameters is as follows:

A. A teacher's beliefs

1. Attitude to mathematics
2. Goals of the teaching of mathematics
3. Educational style (transmission versus constructivist[2])

[2] In the sense of Noddings (1990).

4. Interactional style towards the pupil, to the class (from attitudinal to dialogical) and possibly their projection into the integration of the pupil with people in his/her environment (classmates, parents, grandparents, etc.)
5. Interactional style towards colleagues, school government, inspection
6. A need to develop his/her competences

B. Life experience as a springboard for the teacher's pedagogical beliefs

1. Where it comes from
2. What it concerns
3. What is missing
4. How it is reflected (analysed)
5. Which experience resulted in the shift of the teacher's beliefs

C. Personality

1. Self-confidence in the area of pedagogy
2. Self-confidence in the area of didactic
3. Self-confidence in the area of mathematics
4. Self-confidence in the social area (towards colleagues, school management, inspection, parents)
5. Assessment of one's educational style (does it correspond to reality?)

D. Abilities / competences

1. Pedagogical: management of class discussion, organisation of work, individualisation, creating and maintaining the climate, work with mistakes, patience, etc.
2. Didactic: pupils' motivation, conception of the ontogeny of concepts, relationships, processes, languages, problem, diagnosis of pupils (understanding their ideas), evaluation of pupils, re-education, etc.
3. Mathematical: knowledge of solving strategies of various types of problems, ability to experiment, to effectively use the trial error method, ability to create generic models both procedural and conceptual, posing problem with required characteristics, etc.
4. Social: interaction with colleagues, school management, inspection, parents.

3 Illustration of Parameters

Some of the above parameters will be illustrated by stories of Alice and Matylda.

3.1 Story of Alice

Alice carried out a teaching experiment in grade 3 of an elementary school as a part of the certification process in April 2011. She described and analysed it and in her reflection, she expressed an inner dissatisfaction with the results of her own teaching and initiated discussions with D. Jirotková and M. Hejný. She did not accept their offer that they visit her classes and provide some advice. As an unqualified teacher, she had taught in a transmission way for some years and in the experiment, she was to teach in a constructivist way. She chose the written division algorithm. First, she set the task 57 : 3 = ?. She writes about this part of the experiment:

AL01. "Earlier, I tried to teach children the traditional algorithm of written division. This time, instead of explaining, I had to choose the method of questioning. I asked the children whether they would be able to divide number 57 into 2 numbers divisible by 3. I presumed that they would suggest 30 + 27."

The teacher expected that the children's answers would be in view with the traditional algorithm. She expected the solution: 57 : 3 = (30 + 27) : 3 = 10 + 9 = 19. However, it did not come. Alice went on:

AL02. "The children suggested pairs 51 + 6, 54 + 3, 48 + 9, 45 + 12 (these sums were written in a column below each other)."

Alice seemed to be surprised but she could find the reason for the first division.

AL03. "Some time before that, the children solved 51 : 3."

Alice could not see a way to proceed from the pupils' suggestions to the required relationship 57 : 3 = (30 + 27) : 3 = 10 + 9 = 19. The pupils took over the initiative and found three more divisions of number 57 into addends divisible by 3. Alice wrote:

AL04. "Radka (a weaker pupil) claimed with enthusiasm: '… there is a lot of solutions, it goes one after another, one column grows, the second decreases by the same number.' Many pupils called out at this moment that it grew and decreased by 3."

Alice acquired new experience with children's activity which she gave space to. In her mind, there was a tension. On the one hand, she was glad that the children were active, on the other hand, she could see that the activity did not lead to the didactic goal of the lesson. She found a compromise:

AL05. "For each division, divide both addends by three and add the results."

The children discovered that number 19 resulted in each case. However, Alice added with disappointment:

AL06. "The number of pupils who understood the reason was small."

At the end of her work, Alice evaluated the whole experiment:

AL07. "I was considering the danger that the children would not be able to solve all the problems. Surely, they do not advance so quickly as with the traditional approach when I was writing five-minute tests with pupils on this topic. They had to complete them within a time limit. These pupils would not mostly be able to do so. On the other hand, with this 'freedom' the pupils demonstrated a certain insight into division and into the connection between arithmetic operations, and great excitement for work."

When speaking to the researchers about her experience from the experiment, Alice mentioned two important facts:

AL08. "I was surprised how much the pupils were able to discover, even the weaker ones. I was pleased that they were active."

AL09. "However, I am afraid that I will not be able to organise such teaching and that the pupils' results in tests will be weak. Moreover, I am afraid of the reaction of some more aggressive parents and sometimes of the school management, too."

3.2 Analysis of the story of Alice

The tool of our analysis will be the above set of parameters of the teacher's educational style.

A2.+A3. Goals of the teaching of mathematics and the educational style

Alice can clearly feel the basic polarity of transmission and constructivist teaching of mathematics (AL01), however, she keeps the goal "pupils count reliably and quickly" and the transmission style (AL07).

A4. Interaction style towards the pupil, to the class

Alice tries to proceed in a dialogic way, does not criticise the pupils' answers which do not fit her expectations (AL02), evaluates the pupil's autonomy in a positive way (AL04). She leads the pupil in a non-authoritative way towards the chosen educational goal (AL05).

B1. Origin of life experience

Similarly to most of our teachers, Alice has mostly experienced transmission teaching. As she mentioned several times, she was taught like that when she was a pupil and she has been teaching like that for years (AL01). In the teaching experiment, she acquired one-time positive experience with the constructivist teaching (AL04).

B2. Content of life experience

In the teaching experiment, the teacher saw that constructivist teaching can bring much more joy to pupils than the transmission one. She realised that pupils were capable of generating much more knowledge than she had thought possible (AL04, AL08).

B3. Missing experience

During her university studies, Alice met a constructivist educational style in mathematics, she acquired not only theoretical knowledge, but she also saw example lessons taught in a constructivist way. However, only a small part of it was her personal experience.

B4. Reflection of experience

In the database which concerns Alice, there is no evidence that she tried to rationally reflect on her work. Her reflections are emotional and relate to the failure which she perceives rather fatally.

B5. Experience causing the shift in the teacher's beliefs

New experience was not strong enough to having caused a shift in Alice's beliefs in such a way that she would be able to teach in a constructivist way. She was afraid of moving towards the constructivist style of teaching and soon gave up on the effort to change the transmission style.

C1.+C2.+C4. Self-confidence in the pedagogical, didactic and social areas

Alice herself introduced two reasons for her resignation: (1) she did not believe that she would be able to manage the situation didactically and (2) she was concerned that most pupils would have problems with subject matter (AL09). From the interview, we can infer the third reason: (3) low self-confidence in the social and partially also in the pedagogic areas.

C2.+D2. Self-confidence in the didactic area and didactic abilities / competences

Alice experienced surprise when the pupils proceeded in a different way than expected (AL02). Even though she managed the situation well from the didactic point of view, that is, she found a suitable question which orientated the pupils in the direction of her purposes (AL05), her self-confidence in the didactic area rather suffered, as we noticed several times.

3.3 Story of Matylda

Matylda studied the Faculty of Sciences, Charles University in Prague, the teaching of biology and mathematics for pupils at the lower and upper secondary schools. She graduated in 1992 and then she taught at a secondary school for a short time. In 1999, she started teaching at a small rural school. In 2007–2011, she extended her qualifications for teaching at an elementary school. Within her studies, she elaborated a seminar paper and later also graduation work with the topic 'Probabilistic thinking of elementary school pupils'. She was successful with this work at the national level.

The data are taken from the above two works where dozens of copies of pupils' works are included. Moreover, we have transcriptions of many dialogues and notes from correspondence. Matylda writes about her transfer to the elementary school:

> MT01. "I consider work in a small rural school to be the most enriching from my past practice. It requires perfect preparation, good planning of the lesson, excellent management of pupils. I have taught from grade 3 to 5, I enjoyed the work, the age of pupils was suitable for me."

Life situation required that Matylda left for a big primary school. She explains:

> MT02. "I remained faithful to the same age group (grade 4 and 5), it suits me that the children have not lost their natural desire to discover and learn while at the same time, it is already possible to work on projects with them, discuss things. They orientate themselves in a text, they are more independent."

Matylda did not know how to prepare the teaching experiment. She saw the notion of probability as very demanding for elementary school. Her fears and helplessness intervened in three areas: social, pedagogical, didactic.

MT03. "Children like to boast at home about what they did at school and thus it is necessary to defend the sense of the experiment before the parents, or ask them for their consent that the child is included in the experiment."

MT04. "I was afraid that I would not be able to record all the interesting ideas or that some children would not take part in the discussion because they do not find this way of work suitable for them. I hoped that I would be able to evaluate pupils' answers statistically."

MT05. "I was despondent when introducing the experiment in the class. I thought that I would not be able to explain to children what I want from them. Another important aspect of my fears was that it was an extremely problematic class, heterogeneous as for the social background, nationality and mathematical ability. I had never experienced anything like that."

At the end, Matylda decided to realise the teaching experiment in a written way. She prepared a test/questionnaire for pupils in which she used inappropriately difficult questions and the test was useless for her purposes. Matylda then decided to lead the experiment in a dialogic way. The pupils drew notes from drawing drum, threw dice, discussed things. None of them, however, made any indication of discovering probability notions. Matylda writes:

MT06. "I was desperate that it was a 'lost' lesson. It was pleasant that the pupils spontaneously said that they liked the lesson. At first I did not understand why but I think I know it now – the children rose above the notions which they did not understand and approached the problem in their way. They experienced action and suspense because they forecasted and drew lots."

MT07. "I do not think that the notions of probability would be totally inaccessible to children, but it would require that they throw dice repeatedly, drew lots, etc. and only afterwards would think in a 'probabilistic' way based on their experience."

MT08. "On the basis of the experience from the experiments, I think that it is necessary that elementary pupils really experience the mathematical context in which they work. I recommend a guided discussion so that pupils hear opinions of others and can work with them."

We were interested to what extent the above experience influenced Matylda's educational style after a year. When asked, she said:

MT09. "For sure, this realisation influenced me together with the way mathematics education lessons [at the university] were taught, when we, students, were in the role of elementary pupils and the teacher was in the role of an elementary teacher."

MT10. "I teach grade 4 this year. I try to maximally limit communicating ready-made knowledge and algorithms, I have re-evaluated my way of teaching in many ways. I have put children into the role of brilliant, clever, able DISCOVERERS whom I trust and who can manage everything (I praise, I am exhilarated, I act a bit…). I choose a different formulation of questions – 'What do you think…? How is it that…? And what if…?' I am successful very often."

3.4 Analysis of the story of Matylda

A1. Attitude to mathematics

Matylda, contrary to the most of elementary student teachers, has a qualification in mathematics for teaching the lower and upper secondary school. This fact influences her understanding of mathematics as an exact discipline founded on definitions, statements and proofs. Probability is connected to the definition in her mind.

A2.+A3. Goals of teaching mathematics and educational style

Matylda clearly formulates her effort to teach in a constructivist way (MT10). However, it is not clear to what extent she realises it. When at the end of MT10 she presents questions which she considers to be evidence of her creative teaching, she is limited to the teacher–pupil interaction. There is no mentioning of the discussion within the class.

A4. Interaction style towards the pupil, to the class

Matylda realises the necessity of mutual dialogue of pupils (MT08), but it is not clear from this attitude to what extent the class discussion is directed by pupils.

A6. A need to improve her competences

From Matylda's final work, from her big interest in the solution of the didactic problem which she experienced in the first part of her experimental work, it is clear that she looks for ways to solve it both by the study of literature and consultations, and by reflections of her experience.

B1. Origin of life experience

Matylda's life experience is quite rich and varied. She herself values most the experience which she acquired in a small rural school (MT01) and the experience she acquired during her university studies (MT09). The experiment brings her new experience which she appropriately assesses. (MT08).

B2. Content of life experience

Matylda's markedly different experiences with mathematics education in two types of university studies are in contradiction. In the area of probability, experience from the study of mathematics and from her teaching at a secondary school dominates. This experience makes it difficult for her to solve the present didactic problem with probability at the elementary school.

B3. Missing experience

Matylda misses experience with experimenting, mainly with experiments related to the subject matter at the elementary school (MT05, MT06). She has no experience with a differentiated approach to the class. She prepares the experiment in a frontal way, making no allowance for the heterogeneity of the class.

B4. Reflection of experience

Matylda, to a markedly greater extent than other teachers from our research, believes that she can improve her work by reflecting on experience and its analysis. It can be seen both from her seminar work and from her graduation work (MT08).

B5. Experience causing the shift in the teacher's beliefs

The consequence of the preceding reflection is a shift in Matylda's belief that effective teaching lies in widening pupils' experience which brings both knowledge and motivation (MT08, MT10). This experience became an impulse for a series of other shifts which had been going on for more than a year.

C1. Self-confidence in the pedagogical area

When leaving the small rural school, Matylda had high self-confidence (MT01). It decreased as a consequence of new and unforeseen experience at the new primary school (MT05). Another important decrease appeared when preparing the experiment (MT04).

C2. Self-confidence in the didactic area

Given Matylda's previous didactic success, she had no doubt about her didactic abilities. She attributed problems in the new class to the pedagogical area (MT05). She was taken by surprise by the challenge to carry out an experiment on probability. She could not see how to make this demanding topic available to grade 4 pupils (MT05).

C3. Self-confidence in the mathematical area

Matylda was well aware of her knowledge in the area of probability. She had no idea how to present it to grade 4 pupils. She did not know that both in phylogeny and in ontogeny, the notion of probability is a result of the development whose first stage consists of acquiring experience leading to a syncretic pre-concept of probability (chance, possibilities). Matylda saw her initial failure as a challenge, not as a stroke of fate (MT06). Let us add that this is an inspiration for universities preparing teachers. Only rarely do university students learn about the propaedeutic of concepts, relationships, arguments, processes, situations, etc. they are going to teach.

C4. Self-confidence in the social area

Matylda fears conflicts with parents which might not want their children to be 'guinea pigs' (MT03).

C5. Assessment of one's educational style (*does it correspond to reality?*)

Matylda is rather self-critical and is surprised by the pupils' positive reaction (MT06).

D1.+D2.+D3. Pedagogical, didactic and mathematical abilities / competences

Matylda's key statement in terms of these three parameters is the third and fourth sentence from MT06: "They experienced action and suspense because they forecasted and drew lots." What does Matylda mean by "the children rose above the notions which they did not understand and approached the problem in their way"? We think that when the pupils drew notes with names during the lesson and forecasted what would be drawn, their considerations were totally incomprehensible to her. In Matylda's protocols we can read pupils' statements such as "Jana will be the first to be drawn as she is always the first" or "Eva will be the first because her name has only three letters and such a note is lighter and is lying on the surface". From the pedagogical point of view, Matylda could no longer direct the pupils' work. On the other hand, she gave them enough space for their creativity. From the didactic point of view, Matylda has no other problems prepared to widen the pupils' experience base. From the mathematical point of view, Matylda is captured in her images of probability as a topic of teaching. She has no knowledge of its propaedeutic.

In this respect, we can refer to results of research of J. Kohnová (1995, s. 44). She presents answers of teachers to the question what is the source of help in their education in the areas of a) content, b) methods of teaching. In both cases, self-study comes first and colleagues second. Universities have a score of 15% in the content area and 13% in the methods area. The research is rather old and we can hope that the numbers might have changed with the change of the university education of teachers. Statement MT09 supports this hope.

4 Case Study of Jitka

Contrary to the previous two stories, we have a ten-year database about Jitka which includes her seminar papers, master thesis, rich correspondence between us, with our colleagues and her pupils' parents, many video recordings of her lessons, many interesting mathematical products of herself and her pupils, etc. Moreover, we have been cooperating with Jitka for more than ten years.

4.1 University study

Jitka wanted to study at the Faculty of Education but was not accepted on several occasions. She worked in an after school club and in a kindergarten. After nine years since her secondary school leaving examination, she welcomed an opportunity to teach in grade 4. She was confident enough that she would manage the task. She was taken by surprise that she had to repeat her explanations for some pupils and that not all of them were interested in mathematics. She wanted all pupils to be active. She found partial help in scientific literature, nearly none at her colleagues who, on the contrary, toned down her effort to find more efficient teaching methods. Pupils' positive reactions were her best support. She came to the conclusion that the discrepancy between how she was teaching and how she would like to teach can only be solved by her education.

After eleven years after her secondary school leaving examination, Jitka entered the Faculty of Education with clearly formulated needs. However, at the same time, she was afraid whether her mathematical knowledge would be sufficient. With joy, she stated about her university classes: "… no rules were required from me, my life experience and common sense sufficed. Moreover, I kept getting strong arguments against sceptic colleagues at my school. "

Jitka chose to write master thesis in geometry. She felt her own deficiency in this area but she could also see that in geometry, there was enough space for mathematical and didactic creativity. In the conclusions of her thesis, she writes: "It is strange to consider conclusions at the moment when I have a feeling that many ways are opening to me only now."

4.2 Project

A remarkable result of her thesis, which won a nation-wide competition, inspired us to include Jitka in an international project Comenius. She felt strengthened by these successes and soon acquired a new degree (a kind of postgraduate degree, between the graduate master degree and PhD degree).

In the project, Jitka had to find a cooperating colleague in her school. She writes about it: "… my colleagues did not want to join the project, they feared failure and that none would value it anyway, rather on the contrary." At the end, she persuaded one colleague but it was necessary to encourage and motivate her all the time. (Jitka used the experience which she acquired from this cooperation later when she worked as a teacher educator who also had to encourage and motivate.) When we started working on a new mathematics textbook for grade 1, Jitka was very interested in the material which we were preparing and made arrangements that she could teach in grade 1 at her school and pilot the new textbooks.

4.3 Interaction style towards colleagues

In the summary reflection in school year 2006/2007, Jitka writes (shortened): "I start in grade 1 with which I have had no experience and I am interested in 'our mathematics'. The teacher responsible for all classes in the same grade is an experienced teacher, Maruska. We have 4 parallel classes. Already in September, a tension appears between Maruska and me. The reason lies in different opinions of the teaching of mathematics and reading. Paradoxically, just with my opinions of the teaching of reading, I won experienced colleagues and thus became the authority in our team. Through the success in teaching reading, my colleagues began to be interested in the teaching of mathematics, too. An important moment came when we were teaching 'crossing the ten'.[3] My colleagues did not believe that it was possible to teach addition without the procedure 'crossing the ten'. Maruska claimed that it was nonsense and that I would burn my fingers. When I have so much experience as she has, she continued, I will find it out.

On the one hand, I believed in what I was doing, on the other hand I had doubts. Nevertheless, also some other colleagues tried to avoid crossing the ten procedure and were excited that the children indeed managed doing it and without any explanation.

[3] In most Czech textbooks, addition of the type 7 + 5 is given by the procedure 7 + (3 + 2) = (7 + 3) + 2 = 10 + 2 = 12. This procedure is called 'crossing the ten' and its practising is devoted a lot of time.

This happened at the moment (May) when crossing the ten was in the teaching plan. Maruska had the whole folder of worksheets. She explained to the colleagues that otherwise the children would not understand addition. She decided that I could do it in any way I like but the others would do it properly. Till the end of the school year, the colleagues practised crossing the ten procedure according to these worksheets."

In the following school year when Jitka started to officially pilot our textbooks, tensions between Maruska and Jitka concerning the teaching of mathematics escalated. Two of her colleagues had doubts. Jitka had to face some parents' dissatisfaction. She writes: "I organise meetings with mathematics for them. I have fears and I hope that it will help. A dissatisfied parent is a warning sign for the head teacher and he could ban my next initiative, but I want to teach in this way. For the first time, I ask my colleagues for comparative tests. Tactically, I do not speak about the results which are markedly better in my class before the management of the school. I want my colleagues to trust me that a possible new comparison of the children will not be unpleasant for them." Results of Jitka's teaching were best seen in grade 3 in the nation-wide competition Cvrcek (i.e., a modified Kangaroo competition for the 2nd and 3rd grade pupils). In 2009, from more than 8000 pupils of the Middle Bohemia Region, only 25 got the maximum number of points. Five out of these 25 were from Jitka's class. Her weakest pupil's result was above the whole region average.

Jitka's success in her pupils' knowledge and their approach to mathematics had four important consequences: (1) Jitka has become a teacher educator and with great success disseminates ideas of scheme oriented education in the whole Czech Republic. (2) By her responsive approach to her colleagues, within three years, she won all primary classes teachers of her school for the way of teaching she advocates. (3) She has become a co-author of our teaching materials. (4) She started as a mentor of continual teaching practice of the faculty students and as their teacher at the university. Many students highly appreciate what they are learning from Jitka.

4.4 Analysis

Jitka's case enables us to highlight the interconnection of parameters from section 2. We will focus on the parameters which were illustrated in section 3 only scarcely or not at all.

Jitka studied a secondary pedagogical school and thus had a weaker mathematical education. The university studies which emphasised creativity enabled her to advance in mathematics markedly. She repeatedly experienced joy from solving problems and this motivated her in an on-going way to widen and deepen her mathematical knowledge. (A1, A6)

Already when she came to the university, Jitka felt that the main goals of the teaching of mathematics should be in the development of pupils' thinking, not in the pace of their calculations. During her studies, she was acquiring theoretical knowledge which she immediately used in her class, and thus she interiorised it. Her conviction shifted from the intuitive level to the conscious level, which was supported by her experience and by arguments. By the targeted connection of theory and practice, her teaching became research in action. While teaching, she was able to solve didactic problems which appeared during the preparation of the textbook. (A2)

Jitka's beliefs were close to the constructivist educational style already at the beginning, even though she was taught in a transmission way. She could not use it in her teaching completely because there was a lack of suitable teaching materials. When she first met such materials within the international project and then when trialling the textbooks, her teaching became strongly constructivist. (A3)

Already as a secondary student, Jitka acquired first experience in mathematics education – the teacher asked her to explain subject matter to her classmates and Jitka was successful in it. When she started to work at school as an unqualified teacher, she did not let her experienced colleagues guide her towards transmission teaching. Gradually, she widened her didactic experience in the above activities. Her on-going self-reflection contributed to the speed of her development. Parallel to didactic experience, Jitka acquired pedagogical experience, too, namely, she was better able to understand her pupils. All of them, not only the successful ones, liked her and trusted her. (A4, B1, B2, B4, B5) It is understandable that due to this experience Jitka's self-confidence both in the pedagogical and didactic areas grew. (C1, C2) The comparison of parameter B4 at Alice (without self-reflection), Matylda (occasional) and at Jitka (on-going self-reflection) shows that this parameter has a strong information value in terms of the quality of the teacher's teaching style. The same conclusion has been made by V. Spilková (2011, p. 133).

The social context in which Jitka was working was not favourite to her in the first years. She herself persuaded all colleagues, school management and distrustful parents of the rightness of her way which bears witness of the interaction style of an exceptional quality which guided her. (A5, D4) A teacher's social interaction style strongly depends on his/her social self-reflection and that is strengthened by social success. (C4) At present, Jitka is a confident and modest teacher with an on-going need to reflect, evaluate and improve her work. (B4, C5, A6) Her effort to educate pupils, and not only those from her class, is still markedly dominating her action. (A2)

5 Conclusions

The theoretical considerations above are closely related to the activity of D. Jirotková and M. Hejný who strive to enhance constructivist teaching in the Czech Republic. In the section Case study of Jitka, it is mentioned that the mathematics textbooks written by them and their co-operators for elementary school pupils lead to very good results in terms of pupils' mathematical knowledge and their attitude towards mathematics, provided rules of scheme-oriented education (see Hejný, 2012, this issue) are followed. Similar results are reached in other classes taught in a similar way. The story of Alice, however, shows that textbooks themselves are not sufficient for creative teaching. For sure, Alice has been striving to teach creatively, has been in contact with the theory of scheme-oriented education, has seen lessons taught in this way, but nevertheless, as she writes in AL09, remains faithful to a transmission educational style.

The example of Alice and some other teachers led us to believe that a teacher who teaches in a transmission way has to overcome various obstacles when he/she wants to change his/her educational style. There are two obstacles in the case of Alice: the first is her pedagogical beliefs about the precedence of the pupil's traditional performance (reliability and speed of calculations); the second is a low pedagogical and social self-confidence. Also in other cases which are known to us, the above obstacles play a key role in the failure of an attempt to change an educational style. According to our present experience, we see value polarities as the key factor of the acceptance or refusal of the educational shift. On the one hand, there are pupils and their joy and performance; on the other hand, there is fear of institutional failure and social problems. Alice's statements refer to this:

"I was surprised how much the pupils were able to discover, even the weaker ones. I was pleased that they were active. However, I am afraid that […] the pupils' results in tests will be weak. Moreover, I am afraid of the reaction of some more aggressive parents and sometimes of the school management, too."

Matylda's case is the opposite one. Matylda definitely places the value of the pupils' development at the highest place and she is strongly emotionally influenced by her results when she used new educational procedures. Her need to see the joy of pupils keeps alive her effort to look for and develop educational tools which cause the joy.

The case of Jitka is exceptional. Since the beginning of her career as a teacher, she has been looking for ways to inspire pupils, to give them more than just formulas and algorithms. Once her need met a solid base in her university studies, she immediately took the opportunity and in several years, she has become not only an excellent teacher but also a person who is able to motivate others by her example.

Dozens of investigated cases (three are given in this article) lead us to the conclusion that the quality of the teacher's educational style in mathematics is given by his/her *need to see the joy of pupils from their intellectual growth*. This thesis is in full agreement with Freundenthal's idea *Mathematics is a human activity*.

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Do Student Teachers Attend to Mathematics Specific Phenomena when Observing Mathematics Teaching on Video?[1]

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Abstract: This article describes the results of an investigation into pre-service teachers' ability to notice mathematics specific phenomena in a lesson observed on a video recording. Thirty mathematics education students' written analyses of a viewed lesson were subjected to a selective content analysis. The results of both qualitative and quantitative nature conform with earlier research on pre-service teachers' lesson analyses and, in addition, bring detailed report not only on the participants' ability to notice but also on categories of content-related observable aspects of teaching. The discussion focuses on situating the findings within the framework of pedagogical content knowledge and indicates ways to further link the ability to notice to both teacher development design and effective teaching practice.

Keywords: ability to notice, professional vision, pedagogical content knowledge, lesson analysis, pre-service mathematics teachers, mathematics specific phenomena

1 Introduction

In the present classroom, it does not suffice that a mathematics teacher prepares a lesson well and then enacts it. The role of the teacher is far more complex − pupils should be taking an active part in their learning, with the teacher acting more in the background than traditionally but still guiding their pupils towards the knowledge they are meant to build. Naturally, such ideal lessons cannot be prepared in advance in every detail; rather, the teacher is expected to be able to appropriately react on the spot, to the pupils' suggestions, solving strategies, unexpected situations, etc. The question arises how well teachers and student teachers are prepared for this aspect of their work, how developed their ability to notice relevant aspects of a teaching situation is. While teaching future mathematics teachers at the university, we noticed that our students' written accounts of their teaching practice tended to include general pedagogical comments but rarely comments related to the way they developed their pupils' mathematical knowledge. We have got similar results when student teachers reported on their observation of other teachers' teaching. To focus their attention on features related to mathematics teaching and learning, we asked them to complete a specific task during their teaching practice in which they were to describe three interesting moments from their teaching or from the teaching of others which relate to mathematics learning and teaching. Still, in the last four

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 years at least 40% of the students' responses have been related to mathematics teaching and learning very loosely or not at all. Thus, we became interested in the characteristics (such as the quality and the objects) of student teachers' attention.

2 Theoretical Framework

In research, we find several constructs which try to capture the characteristics of attention from different standpoints: noticing, professional vision, knowing-to, attention-dependent knowledge.

According to Sherin and van Es (2005), *noticing* involves (a) identifying what is important in a teaching situation, (b) making connections between specific classroom interactions and the broader concepts and principles of teaching and learning that they represent, (c) using what teachers know about their specific teaching context to reason about a given situation. Another term describing the same phenomenon is *professional vision* which involves "socially organized ways of seeing and understanding events that are answerable to the distinctive interests of a particular social group" (Goodwin, 1994, p. 606). Thus it makes sense to speak about teachers' professional vision which Sherin (2007) describes as consisting of two distinct, but intertwined, sub-processes: *selective attention* and *knowledge based reasoning*. Teaching is a very complex process. There is a lot going on in every mathematics lesson and the teacher cannot pay attention to everything, certain features stand out for him/her – then we speak about selective attention. Once "the teacher's attention is drawn to a particular event, next the teacher will begin to reason about the event based on his or her knowledge and understanding" (Sherin, 2007, p. 385).

The (student) teachers' ability to notice is important for the development of what Mason and Spence (1999) call *knowing-to*: "Knowing-to is active knowledge which is present in the moment when it is required." They distinguish this kind of knowledge from knowing-that, knowing-how, and knowing-why. Knowing-to triggers the other types of knowing and thus its absence blocks "teachers from responding creatively in the moment" (ibid). While Mason and Spence mostly concentrate on the way knowing-to develops in pupils (e.g., while solving problems), they also touch on educating teachers to be able to know-to:

"We propose that knowing-to act in the moment depends on the structure of attention in the moment, depends on what one is aware of. Educating this awareness is most effectively done by labelling experiences in which powers have been exhibited, and developing a rich network of connections and triggers so that actions 'come to mind'." (ibid)

In the same spirit, Ainley and Luntley (2007) propose the term *attention-dependent knowledge* for "highly contextualised propositional knowledge that is made available by attending to aspects of the classroom situation", that is, the knowledge that enables teachers to respond effectively to what happens during the lesson. It can only be revealed in the classroom.

"It is knowledge that becomes available during the complexity of the progress of a lesson, often in response to instances of pupil activity that could not be predicted on the basis of the teacher's subject or pedagogical knowledge. However, we conjecture that it is attention-dependent knowledge that enables teachers to act effectively in response to what happens during the lesson." (ibid)

We consider the teacher's professional vision part of pedagogical content knowledge (PCK) (Shulman, 1986). For example, Bromme (2008) claims that PCK can also be seen in the ways the teacher "takes into account pupils' utterances and their previous knowledge". An (2004) stresses four aspects of the effective teacher's activity in the classroom which are part of PCK: building on students' mathematical ideas, addressing and correcting students' misconceptions, engaging students in mathematics learning, and promoting and supporting students' thinking mathematically. In order for the teacher to take into account the pupil's utterance and build on his/her understanding, he/she has to notice the importance of this utterance in the first place, put it into the appropriate context, interpret it, and only afterwards use it.

There is an agreement in literature that the ability to notice can be studied (and also developed), among others, by letting (student) teachers analyse video recordings of the teaching of others and/or their own (e.g., Borko et al, 2008; Goffree & Oonk, 2001; Hošpesová, Tichá & Macháčková, 2007; Krammer et al, 2006; Lampert & Ball, 1998; Llinares & Valls, 2009; Muñoz-Catalán, Carrillo & Climent, 2007; Santagata, Zannoni & Stigler, 2007; Sherin & van Es, 2005; Star & Strickland, 2008; Tichá & Hošpesová, 2006). Most of the studies confirm that (student) teachers must learn what to notice. For example, Santagata, Zannoni and Stigler (2007) found out that "more hours of observations per se [...] do not affect the quality of preservice teachers' analyses", and on the other hand, Star and Strickland (2008) claim that the ability to learn from observations of teaching "(either live or on video) is critically dependent on what is actually noticed (attended to)". Blomberg et al (2011) point out that videos of lessons "represent both subject specific and generic aspects of instruction and thus have the potential to activate knowledge of both these aspects". It is this perspective that our study assumes: distinguishing between mathematics specific and generic aspects of teaching, we will focus on the former of the two.

2.1 Research on noticing mathematics specific phenomena

From research studies on noticing, we will present only those which meet two criteria. First, as we observed that our student teachers' attention was particularly drawn to general pedagogical aspects of teaching, as already stated above, we choose studies which specifically investigated whether and in what way (student) mathematics teachers attended to phenomena specific to mathematics teaching (as opposed to phenomena pertinent to teaching any other subject). Second, only work which aims at the types of phenomena which (student) mathematics teachers noticed in a mathematics lesson without any previous training is presented. As most of such studies are of the intervention type, primarily the (student) teachers' noticing of phenomena related to mathematics prior to the intervention will be presented here.

In their larger scale study, Santagata, Zannoni and Stigler (2007) investigated the quality of the analyses of 140 student teachers divided into two groups. The first group of students was asked to observe a video of a whole mathematics lesson and the second one saw parts of the lesson only, in order to reduce the complexity of the teaching situation and focus the students' attention to fewer phenomena. The results were similar in both cases. One of the categories the authors coded for in students' analyses was Mathematics Content. The comments were coded as high quality if they analysed teacher's and pupils' actions in relation to the mathematical content. The analyses were given a score of 1 for this category if they included mostly low-quality comments; a score of 2 if they included a balance of high- and low-quality comments; and a score of 3 if they included mostly high-quality comments. The results for the pre-test were 1.37 for the first group of students and 1.57 for the second and the authors concluded that in "the pre-test, the comments tended to be about general didactic choices and, when the mathematical content of the lesson was mentioned, only seldom were mathematical ideas used directly to discuss the teacher's actions". The score for the post-test was 2.02, 2.05 respectively, and "participants more often used their knowledge of mathematical concepts to shed light on what they observed, or to argue for the efficacy (or lack thereof) of the teacher's choices".

Similar conclusions were drawn in some small scale studies. Star and Strickland (2008) conducted a study with 28 student mathematics teachers. On the basis of an expert analysis of a mathematics lesson, they created a set of questions (multiple choice, yes/no and short answer questions) which the students were asked to answer from their memory after seeing a lesson on video. They found that the students did not enter mathematics teaching courses with well-developed observational skills and specifically that "preservice teachers were not particularly observant of more substantive features of classrooms, particularly mathematical content" and this remained true even after the intervention course. When content was noticed, the students "tended to comment only about whether the content was presented accurately and clearly and/or to provide a chronological description of what the teacher wrote on the board during the lesson". Star and Strickland conclude that the students "largely did not notice subtleties in the ways that the teacher helped students think about content".

Alsawaie and Alghazo (2010) conducted a small scale study in which 26 student teachers took part. They found out that prior to the video based course, the students' analyses of a mathematics lesson tended to include chronological descriptions of most what happened in the lesson with no interpretation and with no identification of noteworthy events. Similar results were reached in a small scale study by Sherin and van Es (2005). Both studies report a significant change after the course in the choice of noteworthy events and the way the participants saw them – they used fewer evaluation comments and more evidence-based comments.

The above research shows that (student) teachers, without guidance, notice general educational phenomena or phenomena of classroom management rather than phenomena related to mathematics in the video recordings of mathematics teaching. However, they do not go into much detail into what kind of mathematical phenomena were noticed and which ones were neglected. Thus, in our research, we have decided to focus our attention on the mathematics specific phenomena only. This also enabled us to reduce the complexity of coding criteria which some of the authors report (e.g., Santagata, Zannoni & Stigler, 2007) and which we, too, encountered when creating a code system that would encompass all elements of the discourse found in student teachers' written comments on video recorded mathematics lessons.

2.2 Mathematics specific phenomena

None of the studies above includes a unique category for mathematics specific phenomena as we see them. For example, in Star and Strickland's (2008) work, it would span three categories: Tasks, which refer more generally to activities pupils do (and thus include both generic and subject specific aspects of teaching), Mathematical Content, which includes representation of the mathematics involved (graphs, equations, tables, models), examples used, and problems posed, and Communication, which includes pupil-to-pupil as well as teacher-to-pupil communication aspects, such as questions posed, answers or suggestions offered, and word choice. Santagata, Zannoni, and Stigler (2007) code for five dimensions, three of which overlap with mathematics specific content: the Mathematics Content (for coding all the comments – "comments that did not mention the mathematics presented in the lesson were coded as low quality; comments that analysed teacher's and students' actions in relation to the mathematical content were coded as high quality"), Student Learning and Critical Approach dimensions. Finally, Van Es and Sherin (2010) used dimensions to code for each comment, and one of them was Topic, which included Mathematical Thinking, Pedagogy, Climate, Management, or Other. In our study, the mathematics specific category would span Mathematical Thinking, which refers to mathematical ideas and understandings, and the part of Pedagogy which refers to techniques and strategies for teaching the subject matter.

By *mathematics specific phenomena* (or MS), we will mean the phenomena that could be observed, explained, inferred or interpreted in relation to either mathematical or didactical issues pertaining to the teaching or learning of mathematics as opposed to other subjects. Thus, MS category can be seen as a part of professional vision of a teacher of mathematics as opposed to a teacher of other subjects. Further clarification of this concept will be made by examples in the analysis below.

3 Methodology

We focus on the following research questions:

1. Which MS phenomena do student teachers attend to and which do they miss when not given a specific focus area (selective attention)?
2. How do they interpret these phenomena (knowledge-based reasoning)?

The data consists of written analyses based on a video recording of one particular lesson, namely, an Australian mathematics lesson from Grade 8 from TIMSS Video Study 1999. We consider the lesson to be reasonably rich in generic and subject didactic content (Blomberg et al., 2011) and thus it offers a solid base for our study. The topic was the division of a quantity in a given ratio. The teacher starts out introducing the topic, providing pairs of students with wooden blocks and leading the class through a sequence of graded modelling tasks to arrive at the concept of division of a quantity in a given ratio. Following this introductory part, the pupils are asked to create their own problem based on dividing in a ratio and have their partner solve it. Then the teacher draws her pupils' attention to practical real life applications of this type of problem and asks them to create "a story" based on the problem they had made up previously. From the pupils' elicited answers it is evident that the task was not clearly stated and the teacher tries to clarify the situation. Then the teacher quickly shows another way to solve a problem by using fractions and asks pupils which method they prefer. Individual textbook practice assignment follows, and the lesson's last part is spent on the initial stage of an investigative activity with "Smarties" sweets, focused on statistical topics, such as frequency and percentage. Further relevant aspects will be described in the text below.

Thirty students,[2] future mathematics teachers, who enrolled in mathematics education (here ME) course at the authors' department between 2008 and 2011, participated in this study. Twelve of them were students of the first semester-long course in ME (Group A), 13 were students from the second, continuation, course and 5 students from the third course (Group B). All of them had studied mathematics for three years at the university level before taking the ME courses. Their teaching practice takes place after the first course in ME, thus it can be assumed that only Group B students had had between 8 and 16 mathematics lesson observations and between 12 and 24 lessons of their own teaching (to pupils 12 to 19 years old). Both groups had participated in a pedagogical-psychological practicum in which they observed lessons of different subjects in schools for one semester one day a week and had seminars with a psychologist and a general educator speaking about their observations.

In the study, students were asked to freely reflect on the lesson at home; they were instructed to watch the video or parts of it as many times as they needed,

[2] We use the term pupils to refer to pupils in the recorded lesson and the term students to refer to university students, future mathematics teachers.

pausing, rewinding and forwarding it at their leisure. No expected length or structure of the text were specified. The students submitted their analyses in Word files or via Moodle.

At the beginning of the series of three ME courses, the students had no experience with analysing videos, however, as the task was set within the ME course, we presumed that they would naturally tend to notice or write about MS aspects of the lesson rather than the generic ones.

Within the ME courses, the first of the authors quite frequently uses analyses of video clips as class activities to illustrate a teacher's approach, pupils' reactions, teacher-pupil communication, etc., or to elicit the students' views of mathematics teaching, stimulate their thinking about teaching a particular concept, etc. However, the development of the ability to notice is not the main focus of the courses.

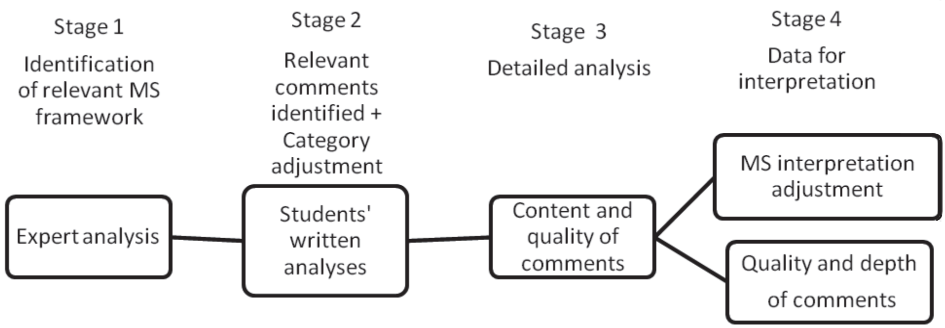

Figure 1 Data analysis. The diagram of the data extraction and organisation

Figure 1 represents the entire process of data extraction and organisation which ended with a theoretical saturation (Strauss & Corbin, 1998). Similarly to Star and Strickland (2008) and Blomberg et al. (2011), we carried out an expert analysis of the mathematics lesson in question in the first stage. An expert analysis framework for this particular video recording was drawn based on multiple viewings and was subject to various revisions of three independent ME experts. The process of this analysis was not meant to simulate the context in which students were writing their analyses (as we assume that the majority of them did not resort to multiple viewings, for example) but rather aimed at identifying as many MS significant phenomena as possible in order to encompass the wide range of potential responses. There were 11 main coding categories identified by and commented on in the expert analysis.

In the second stage, the students' written analyses were coded for the expert analysis categories in the Atlas.ti software (while scanning the text for comments on any additional MS phenomena). There were some minor adjustments made to the categories: the category *Teacher's mathematical error*, which was initially spread across other categories, was added to the framework, and the category *Process vs. concept* was omitted as no data were coded in the students' analyses as relevant.

 After these adjustments, the category system was discussed in a group of researchers so that inter-coder reliability was ensured.

The data extracted by applying the system of the 11 categories would be significant in telling us *whether* and *what* the students noticed and chose to report on. However, in order to get a deeper understanding, we decided to analyse the categories further, hoping that such analysis would lead to answering our *how* question. Thus, in the third stage of the analysis, we identified codes which showed content or depth of individual remarks within each category. Some of these codes were already identified in the expert analysis, some resulted from the content of the students' text. The nature of such sub-division proved to be in some cases disjunctive (e.g., students either interpreted the last activity in the lesson as aligned with the lesson topic or not) and cumulative in others (e.g., a student could report a *specific* case of dealing with a pupil's response as well as comment *in general* on the teacher's use of pupils' feedback from activities at the same time). Sometimes the subcategories were established based on specific interpretation of a situation (e.g., whether students described the ratio-quantity segment of the lesson as teacher explanation or student discovery – an interesting example of contradictory interpretations of an identical situation captured on video).

The main purpose of creating the code system was to get as much relevant insight into the qualitative aspect of the comments as possible. Thus, for example, the subject of elaboration was sidestepped as non-relevant for our study in some cases (e.g., codes in category *Reinforcement of previously learned concepts* offered natural opportunities for simple description, especially as the phenomenon was a marginal one in the lesson) while in others a simple description of a situation in the students' analyses resulted in the loss of MS aspect of the comment and thus was not used as input in our study (e.g., when a student simply states that "in the next stage, the teacher expresses given parts as fractions", the recount itself lacks any MS dimension). In fact, we began to notice that, apparently, not every phenomenon from the framework was being viewed by the students from the MS perspective. In consequence, in the final stage of the analysis, we decided to consider eight codes as not directly concerned with an issue specific to the domain of ME or mathematics. For example, if a student described or even praised the use of blocks in the lesson, *without* referring to their modelling role, we did not assign this comment a MS aspect. In the same way, the set of comments made about the teacher's dealing with pupil responses *in general* (e.g., "The teacher answers her own question instead, leads the pupils to an answer while she should let them formulate their own answer.") while specific cases of pupil response and/or the teacher's reaction with specific mathematical content were coded as MS (e.g., "Pupils were further asked to come up with a story about why we should divide 210 Smarties in the ratio 2 : 5. This caused some difficulties, because they answered that they wanted to divide them by colour or among 7 people."). After much deliberation, we also chose not to include general comments about real-life connections, motivational aspects of the pupil-posing activities and non-elaborated descriptions of the activities. By omitting these non-MS comments, we extracted *data adjusted for MS dimension*.

The detailed analysis also enabled us to code for the use of notions and terminology frequented in the theoretical groundings of the ME courses as well as for the student's propensity to criticize and/or offer alternatives to the teacher's MS related actions.

4 Results

The students' written analyses differed in length, from a single paragraph to two pages (from 76 to 1185 words, with the mean value of 460 words). The ability to notice MS phenomena can be measured, for example, by the number of categories that appeared in individual analyses. None of the students considered the maximum 11 categories, while one commented on 10 categories, and the lowest scoring four students mentioned one category only (each a different one). Fig. 2 divides the analyses into four subgroups. It is noteworthy that 18 students (60%) reported (from the MS perspective) on less than a half of the 11 identified categories.

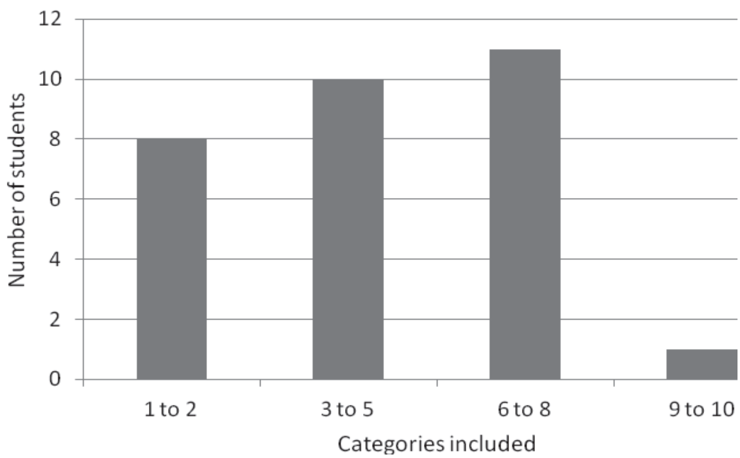

Figure 2 Number of categories noticed in analyses

Figure 3 depicts which particular phenomena were most or least noted by the students. We can see that the *Pupil problem-posing activity* received most attention (albeit of only 63% of total students). It was rather prominent in the lesson, as it was comprised of two stages and the class-time devoted to this activity was considerable. What we found from the analysis, though, is that its benefits were attributed as often to motivation and classroom-management (12 comments, e.g., "Pupils are only engaged in the lesson when they are given the task to create their own problem and their own story") as to epistemological issues (12 comments, e.g., "Thinking about own stories seemed to be very useful: why should I divide something in a ratio? […] Creating a story, the pupils had to make various connections and grasp the meaning of what they are doing and calculating."). One reason for this could be the fact that pupil problem-posing is not an activity that is traditionally used in Czech classroom practice, and many students commented on the motivational strength of "novel" or "unusual" activity, because they simply viewed it as such. Ten out of the nineteen students who commented on this category offered an alternative to the activity management. Six students made neither a didactic nor a general pedagogical comment at all.

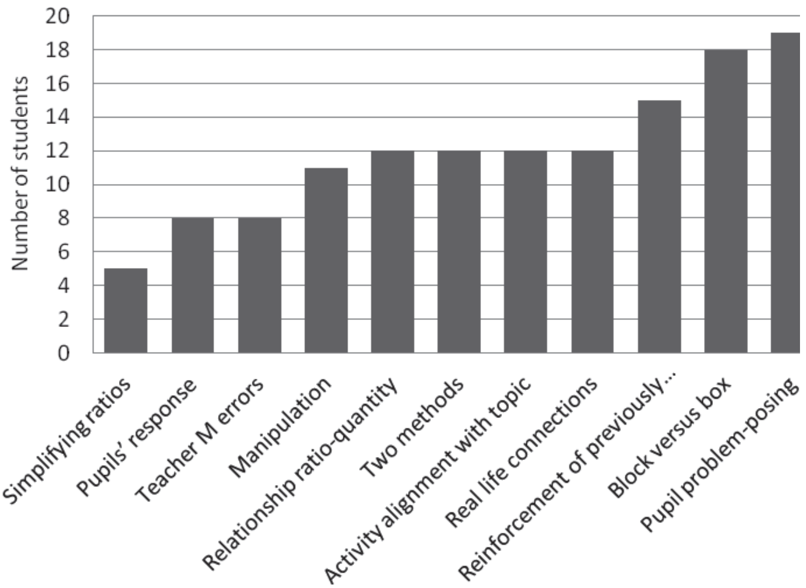

Figure 3 Individual categories noticed in analyses (adjusted for MS)

The other strongly represented category (60% of total students) was *Block versus box*. While blocks are counted as separate items, the empty boxes stand for a certain unknown number (or amount). Each must contain the same number (or amount). The letters *a*, *b* in the ratio *a* : *b* stand not only for a certain number of things but also for groups of (or boxes full of) things.

We used the following codes to measure the ability to notice the separate models in the sequence that the teacher uses to inductively introduce the lesson's topic:

Model 1: Each block represents a counting unit *and* a dividing unit/"share" (e.g., divide 12 boxes in the ratio 5 : 7).

Model 2: Each block represents a counting unit but *not* a dividing unit (e.g., divide 12 boxes in the ratio 1 : 2, "make them *equal piles*").

Model 3a: Each block represents a dividing unit (a share) and model 1 is applied ("How many boxes do you need if you divide 1 : 3?"). But now the idea of a box with some content is implied.

Model 3b: Each block represents a dividing unit (a pile, box, share) and is assigned (filled with) the same number of counting units.

Only a third of the students noted the transition between model 1 and 3b and in doing so they did not elaborate on the in-between stages. Seven of the students focused on the extension of the model 3b onto different types of counting units. Only one student distinguished the entire sequence.

The least noticed phenomenon was the one concerning the simplification of ratios. The opportunity to discuss simplifying ratios and/or the connection between ratio $a : b$ equalling $ax : bx$, where x is the amount of counting units in a share, comes up in the lesson on at least six occasions. Yet, the teacher seems to be avoiding the issue by performing the simplification herself whenever the need arises but neither addressing it directly nor mathematically explaining why she does (or does not) do so. Only 5 students noticed this concept at all, and only three of them commented on the teacher's not pursuing the topic. We can claim that this was the MS category that required the most advanced, "read between the lines" ability to notice.

Noticing pupils' responses and/or the teacher's work with pupils' responses also scored a low score (7 students reported on one or two particular situations, 1 student on six). In the expert analysis, we identified a minimum of seven major observable cases of teacher's handling a pupil response in a MS context.

The division of a quantity in a given ratio is introduced in the lesson using the model of cubes and boxes. This should help pupils to build an image of the whole process. The pupils first work with cubes and create ratios such as 1 : 2, 5 : 8, etc. Then they work with empty boxes. When solving problems, they are asked to first model the situation and only then to calculate. The aspect of manipulation was mentioned by 16 students (53%), however, only 11 (37%) mentioned the relevant process of modelling.

An interesting result arose in the case of interpretation of the final class activity (*Activity alignment with the topic*). Some students interpreted it as aligned with the topic of division of ratio, and some noted the switch to statistical data analysis. This case of contradictory interpretations can be accounted for by at least two factors: (a) the lesson was unusually long for Czech standards (students commented on this), so some students may have recognized the textbook-practice activity as a planned final stage of a unit and naturally viewed the remaining time as time for introducing a new topic, and (b) the lesson was long for critical viewing and it is likely that towards the end students' concentration dropped and they *assumed* the lesson's cohesiveness and the final activity to be working further with ratios, without taking the opportunity to examine the activity hand-out properly.

Finally, let us look into how students comment on what they see as negative (MS) points in the lesson, and whether they offer an alternative. Altogether 170 comments were coded as MS adjusted ones (76 were made by Group A students and 94 by Group B students). As a rule, in MS adjusted comments, a critical remark was accompanied by suggesting an alternative (or a correct mathematical answer in case of commenting on the teacher's mathematical errors). We found out that 28% (47 comments) of the total of MS comments were of a critical nature, including comments related to the teacher's mathematical errors (10 comments). Didactical alternatives for the teacher's action were included in 21% (37 comments). The alternatives regarded mostly a more efficient realization of activities (e.g., "The teacher gives her pupils time to derive the rule, in the end, though, doesn't let anyone explain, and conveys it to them herself."), including important mathematical aspects of the topic (i.e., the simplification of ratio), more elaborate work with pupils' responses, and alignment of the final stage of the lesson with the lesson's main topic.

Twelve students did not make any MS related critical remarks; on the other hand, four students gave an alternative in over one half of their comments. Figure 4 shows the representation of alternatives in all MS adjusted analyses. Naturally, the frequency of alternatives is, to a large extent, determined by the specifics of the lesson analysed.

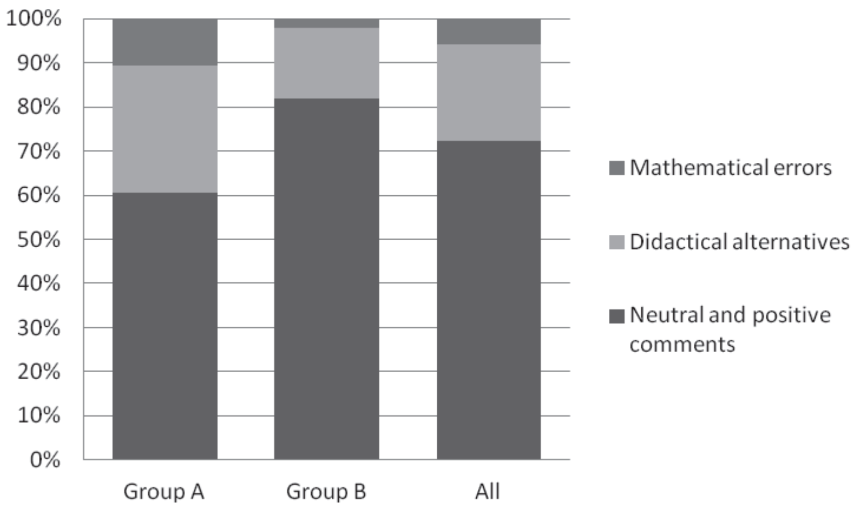

Figure 4 Neutral and positive comments vs. alternatives

When assessing the students' use of notions and terminology frequented in the theoretical groundings, the analysed text contained only five remarks with reference to ME theory, each made by a different student. These referred to the theory of generic models, which is a concept development theory (Hejný, 2003).

Our investigation of differences between students in the first course of ME (Group A) and students of the second or third semester courses (Group B) did not yield any conclusive results. In terms of the ability to notice, the different MS phenomena, two categories (*Teacher's mathematical errors* and *Simplifying ratio*) stood out, however, we do not believe that the amount of data involved is enough to assure statistical significance. Perhaps one distinction between these two groups was apparent and seems to be of interest here: we noticed that the propensity to provide didactical alternatives and correcting teacher's mathematical errors was stronger in Group A (where 40% of all MS comments were such alternatives, compared to only

18% of total comments in Group B). Figure 4 shows the difference between the two groups. This difference may be an occurrence particular to our group of participants. It can also be conjectured that students with hands-on teaching experience (i.e,. student-teaching) are more empathetic with the observed teacher. Nevertheless, such hypothetical causality would need to be explored further on a larger set of data.

5 Discussion

It is important to note here that the studied analyses were rich in comments concerning non-MS (i.e., especially generic and classroom management) issues. Our study was conducted from a perspective of the belief that perceiving classroom situations mathematically is an important aspect of teaching practice. Focusing on MS phenomena enabled us to look more deeply into the nature of what in MS phenomena is being noticed or omitted.

It has been widely acknowledged that a teacher's mathematics knowledge is only one of the pre-requisites for teaching practice conductive to mathematics learning. Measuring how much our students have been able to observe informs us on how aware they are of the complexity of the mathematical skills required of a successful teacher (in other words, how developed their PCK is). This study confirmed the results of research presented above in that student teachers often neglect the mathematical aspect of teaching situations. In particular, the following elements of instruction can be identified as strongly dependent on PCK:

Introducing a topic

In the lesson at hand, the first (inductive) stage required a careful examination of the target topic (i.e., a chosen set of concepts)[3] – dividing a quantity in a given ratio – in terms of its inner structure and characteristics, considering the "building blocks", e.g., the concepts of ratio, divisibility (a consideration included in the category *Relationship between ratio and quantity*), equal parts and irreducibility (as in the category *Simplifying ratios*). The individual models are exemplified and enacted using manipulatives. Yet, from the students' analyses it perspires that this carefully devised series of graded tasks and examples in the inductively led introduction of the topic itself went unnoticed by the majority of students. Although it can be assumed that they themselves have the knowledge to perform the division of a quantity in a given ratio (that is, the content knowledge), they do not appreciate the process of breaking the topic down and examining the fine points (their PCK is insufficient in this area). They only notice and comment on what they see on the surface, i.e., the fact that the teacher uses blocks and boxes. There is no significant difference between the two groups of students. The reason might be that the students do not

3 It is clear from the teacher's lesson plan, which is available for the lesson in question, that the introduction of new material is the bulk of her preparation and that she examined the sequence of models in depth.

have enough experience with preparing an introduction of subject matter based on the inductive method (which has a pedagogical implication for their ME courses).

Choosing relevant instructional activities and providing various representations

The other level of mathematics specific awareness is the target set's place in the structure of mathematical concepts (most importantly, but not exclusively, in relation to curriculum, i.e., considering questions like these: What do my pupils know already, how does relate this new topic to that knowledge? What do I need to review/activate, focus on? Are there other ways of solving the problem? What other concepts might come up during the lesson and in what context? Which ones do I want to explore? How do my activities relate to future learning?) as well as its applications in other subjects or areas of real-life experience (Why is it important for my pupils to be able to do this part of mathematics? How does this mathematics relate to their personal experiences? Which applications offer a new way to represent the concepts?). In our specific lesson, this knowledge comes through most prominently in categories *Pupil's problem posing*, *Two methods*, *Teacher's reinforcement of previous knowledge*, and *Activity alignment*. Again, if we look at the students' written analyses as indicators of this MS knowledge applied to this lesson, we see that although these categories represent the more noticed ones, they are ignored by the majority of students (with the exception of *Pupil's problem posing*, as discussed earlier).

Working with pupils' answers

The ability to understand, interpret and react to pupils' responses and notions is highly dependent on PCK. This is, of course, on two levels, one of them is to notice the answer and one is to notice the teacher's handling it, but they are actually in one, as a non-descriptive remark on the first leads automatically to handling it in most of the cases we observed in our analysis. Our students' low performance on this particular subject of attention aligns with the results of others (e.g. Star & Strickland, 2008). It has been suggested by Gall and Acheson (2010) that noticing the pupils and tuning into their learning processes is one of the further stages on a teacher's learning path.[4] Spangler (2011) stresses the importance of training student teachers in reading and reacting adequately to pupils' responses and errors. Our own study results further confirm these points.

Next, let us summarize some other results from the presented study. In their above research, Santagata, Zannoni, and Stigler (2007) found out that the participants' comments in the pre-test were mostly positive and only after the course, they "assumed a more critical approach when watching the lesson. […] In the post-test, participants re-evaluated some of their observations. They noticed some contradictions in the teacher's actions. They reflected on what they observed, discussed

[4] Of course, the nature of the video recording may have also contributed to the lack of students' attention to pupils' learning – the video-camera mostly focused on the teacher and the class as a whole.

possible problems, and often proposed alternative actions." In our research, we got contradictory results. As shown above, our students were quite critical to the teacher's action and proposed alternatives, and more strikingly, the less experienced group of students did so more frequently than Group B. This can be perhaps explained by a certain level of compassion towards a teacher and a reluctance to criticize in Group B (as the students can better relate to the teacher as their future role), or by being able to stress the positive aspects of viewed material.

The authors who use videos in their courses (some references given above) claim, among others, that the video is a way to connect the theoretical knowledge taught in the courses and practice. However, we were disappointed to see that our students in Group B did not use their theoretical knowledge taught in the ME courses for the description and interpretation of the lesson observed. This has an important pedagogical implication for the ME courses – tasks must be developed which explicitly ask for the description of some pedagogical situations in terms of the theoretical concepts. It seems that the connection of the theory and practice is far from straightforward.

Finally, as stated above, both groups of students had had an experience with analysing observations of lessons from the point of view of generic aspects rather than mathematical. Group B students also took part in a teaching practice in mathematics during which they were provided (mandatory) opportunities to observe experienced mathematics teachers and to reflect on these observations. But, as we can see from our study and the study of others, too, this learning is far from self-evident. Star and Strickland (2008) point out that student teachers often observe lessons as learners of mathematics, not as mathematics teachers: "The kind of observing that one does as a learner typically concerns the comprehension of the presented material (e.g., Do I understand what was just said? Does the mathematics make sense to me?) and does not prompt the observer to think deeply about the teaching and learning process more generally." Thus, we believe that even the 'ordinary' ME courses not only the ones which are organised around videos should include activities which aim to develop student teachers' ability to notice MS phenomena.

To sum up, in this study we devised a method for measuring mathematics education students' ability to notice and comment on *mathematics specific phenomena*. This method is based on analyzing students' written reports on a specific lesson. Although we are aware of the limitations of the scope of our study for making significant conclusions (the data is not extensive enough and the factors involved are too many), in terms of student experience with ME environment and/or exposure to ME course material, we hope these results will inspire future investigations. Perceiving situations mathematically can be a matter of training (van Es & Sherin, 2002). It is important when the students are in the role of a teacher: they should be able to react in such situations without losing track of the mathematics behind it. For example, being able to introduce a topic effectively, choose relevant activities and analyse pupils' misconceptions or work with pupils' approaches to problem-solving.

The research presented in this paper forms part of a wider study aimed at the student teachers' ability to notice MS phenomena when observing a video recording of a lesson or its segments. Future analysis of data will be conducted in the effort to shed light on individual students' development of ability to notice these phenomena across time and ME course attendance, on the effect watching selected segments (rather than a whole lesson) may have on commenting on MS, or on identifying general trends in the nature of MS aspects that are noticed or missed by pre-service mathematics teachers. A comparative study on in-service teachers could also bring relevant findings and deepen the understanding of this issue as well as provide important evidence to guide the design of pre-service teacher development programs, and potentially help link the ability to notice specific phenomena to effective teaching practice.

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Dispute in Mathematical Classroom Discourse – "No go" or Chance for Fundamental Learning?

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Abstract: Disputed points in school mathematical discourse are rarely discussed. If mathematics is seen as a "ready-made" discipline which teaches dealing with right and wrong only, mathematical questions which require debating do not play an essential role. Looking at the real teaching practice shows indeed quite a different picture: Mathematical questions which require debate and justifications appear in the course of mathematical interaction in different places and are dealt with in different intensity. In this article we present small group discussions within the project *Probing and Evaluating Focusing Interaction Strategies in Elementary Mathematics Teaching* (*ProFIT*). The discourse is based on disputed points which turned out to be a promising way of supporting mathematical oriented discussions, a deeper understanding of ambiguous learning situations and the chances for fundamental learning.

Keywords: dispute, discursive learning, focusing interaction strategies

1 Introduction

Common reform efforts emphasise mathematics as science of patterns (Devlin, 1994; Wittmann, 1995). To communicate in mathematics a system of symbols is needed. According to Steinbring (2005), it is impossible to communicate about mathematics in a direct manner. Mathematical signs and symbols do not directly refer to real objects, but symbolise the structure, relationship, patterns, arrangements, connections, etc. between different mathematical objects in an operative way which lies in the special nature of mathematical signs and symbols (Steinbring, 1999, p. 13). The dilemma is that mathematical signs and symbols are often used in many different ways and thus gain an ambiguous status. A mathematician has several convictions and deductions in mind for dealing adequately with the defined symbol systems and its derivations in different situations. However, as pupils at elementary school are in a learning process, they begin to construct their own understanding of mathematical signs and symbols and how these are used correctly in different mathematical situations. Certainly – like in other subjects – young pupils do not always use the signs and symbols in the conventionally correct manner from the beginning. A mutual discourse gives them the opportunity to (re-)phrase their interpretations to each other. Of course, their interpretations are sometimes inconsistent or contradictory and lead to dispute or disagreement among the interlocutors.

Recent research reports identified dialogical learning as essential to the teaching and learning of mathematics (e.g. Sfard, 2000; Steinbring, 2005; Nührenbörger & Steinbring, 2009). However, the teaching discourse cannot ensure fundamental learning on its own. The way of teaching and the special teacher's role in the discourse essentially affect how profound dialogical learning will be for the participants. Through analysing discourse, Wertsch and Toma (1995) distinguish between univocal discourse (one-way communication from a sender to a listener) and dialogical discourse (constructing meaning by joint communication). While many studies focus on the teachers' proficiency in the sense of gathering mathematical know-how and on the educational knowledge of mathematics teachers (e.g. Kunter et al., 2011), further research is needed to understand the role of teachers in supporting the discursive negotiation of mathematical meanings and, by that, supporting the pupils' relational learning by classroom interaction.

2 Perspectives on Mathematics: Mathematics Requires Construction as well as Interpretation

Both at school and in science, mathematics displays a highly complex system of signs and symbols. Mathematicians use these signs and symbols routinely and naturally – usually without thinking explicitly about its "correct" use. This is not meant to point to deficits of young pupils' mathematical activities compared with mathematicians' research work. First of all, this should make explicit the specific epistemological challenges of understanding mathematical knowledge. Duval explains this position as "paradoxical character of mathematical knowledge":

> […] there is an important gap between mathematical knowledge and knowledge in other sciences such as astronomy, physics, biology, or botany. We do not have any perceptive or instrumental access to mathematical objects, even the most elementary, […]. We cannot see them, study them through a microscope or take a picture of them. The only way of gaining access to them is using signs, words or symbols, expressions or drawings. But, at the same time, mathematical objects must not be confused with the used semiotic representations. This conflicting requirement makes the specific core of mathematical knowledge. And it begins early with numbers which do not have to be identified with digits and the used numeral systems (binary, decimal). (Duval, 2000, p. 61)

However, especially these interpretations, which often stay implicit in the discipline of science, demand cognitive processes of mathematical understanding which stay at first concealed for the learning pupils. Therefore, in the context of primary school, teaching cannot be based on ready-made mathematics that is transmitted to the pupils as a recipe for using numbers. For pupils, mathematical signs and symbols become an independent requirement of construction and interpretation.

Interpretations are not steady but vary depending on the situational context and the interpreting person. That means different interpretations can exist at the same time, which have to be weighed against each other. Especially at primary school, this is problematic, as many different visual representations are used and various concepts are formed. Therefore, an *empirical* or a *theoretical* view (Steinbring, 1994) towards the mathematical object can be held. The empirical view is related to the concretely perceived objects and their descriptions. The theoretical view does not see mathematical objects as things but mathematical relations are built through them.

2.1 Features of interaction at school about mathematics

With this perception of mathematical concepts, *interaction* at school *about mathematics* makes a difference. The pupils themselves have to participate actively in the interaction and they have to verbalise their interpretations. That means the teacher has to be open towards the pupils’ personal constructs of interpretation and attempts at reasoning. Wood (1994, 1998) characterises this teaching behaviour as the interaction pattern of *focusing* and distinguishes it from the interaction pattern of *funnelling*. She points out the importance of involving the pupils actively in the interaction process, making the mathematical content as well as the communication itself accessible for them, asking them for mathematical explanations and reasons and sharing their ideas during the discourse. The pupils’ different points of view become visible, because of this requirement for the pupils.

In such an interaction, the teacher does not intervene in a controlling or regulating way and the children will not take over the teacher’s view immediately. The teacher explicitly asks the pupils to explain their own views. By explaining mathematical signs and symbols, the pupils refer to “objects/reference contexts” which are in part very different from each other. So it is important to understand that – in our context – the concepts “sign/symbol” and “object/reference context” do not stand on their own in the discourse. The concepts “sign/symbol” and “object/reference context” are, metaphorically spoken, certain means of discursive negotiation of meaning and they do not directly provide the meaning in question, but they serve the elaboration of the epistemology of mathematical knowledge in the social interaction context (Steinbring, 2005, p. 12ff.).

Of course, pupils at primary school are not always able to express themselves decidedly and explain mathematical patterns and structures precisely. The empirical research of Miller shows that “children, who […] are only 7 to 8 years old and older, know basic cognitive and linguistic-communicative techniques and argumentations and are able to consider their own normative point of view or parameter of value and, at the same time, the one of the opponents in argumentations” (Miller, 2006, p. 47, translated by the authors). Consequently, the statements in mathematical classroom discourse do not underlie this objective indefeasibility which the discipline of science, being formally abstract, allows mathematics. Instead, mathematical objects underlie an ambiguity which provides numerous links for discussions in lessons if the teacher is aware of this ambiguity and introduces it as a constitutive element in his or her classes. Already in 1990, Voigt refers to the tension between the childlike endowing of meaning and the teacher’s didactic-curricular motivated interpretations referring to mathematics. But how is it possible to realise that this endowing of meaning as well as the teacher's interpretations referring to mathematics become subject of the interaction?

2.2 Fundamental learning by disputed points in school mathematical discourse

In this article, we present the possibility of emphasizing the participants' different interpretations in the interaction. Voigt stresses the importance of enduring and eliminating competitive ambiguity in classroom interaction (Voigt, 1990, p. 308). A focused interaction, related to the differences and relations between ambiguous objects and the partly competitive interpretations, can lead to dispute. If the interaction does not cease, it provokes the pupils to discuss the mathematical justified disputed points and may be called "focusing".

It is a challenge of such an interaction to perceive and realise the crucial point: "Collective argumentations require perception of inter-individual coordination processes by the interlocutors and they are an attempt to develop collective solutions […]" (Miller, 1986, p. 24, translated by the authors). If the disputed point is not recognised, it cannot lead to fundamental learning processes. Referring to Miller, "fundamental learning processes" lead to new structural (social-)cognitive problem solving and to a progressive and more adequate recognition of a higher tier concerning the world of nature, the social world and the world of the own inner being (1986, pp. 9–10). *Discursive contexts of discovery about new beliefs and new knowledge* (Miller, 1986, pp. 246–341) are the basis for fundamental learning in collective argumentation.

> A discursive context of discovery can be identified and located as the network, constituted by collective argumentation in which possible thoughts and partial arguments develop, that mediate between thesis and antithesis and possibly establish a consensus between both. […] the corresponding discourse processes are not at all arbitrary. They depend on how the participants proceed for gaining a joint understanding about what the question of debate is in their dispute. (Miller, 2006, pp. 216–217, translated by the authors).

According to Miller, a crucial point "is not usually a generic normative utterance, that means utterances of the type "In general, one should…", but singular normative utterances of the type "In situation *s* person *x* should not have done operation *y* but operation *z*". The normative elements of a daily life context or of a general socio-cultural system of values are normally not controversial issues, but, generally, the application of collective moral codes in concrete cases of conflict is controversial." (Miller, 2006, pp. 45–46, translated by the authors)

However, in mathematics we do not have a moral code, but a sophisticated system of signs and symbols, which are subjected to manifold relations regarding to conventions and deductions. Through this, a reformulation of Miller's proposition may look like this: "The student *x* is more likely to make the interpretation *y* than *z* of mathematical signs and symbols. Meanwhile, he or she should refer to the object *r*." Without a doubt, this refers to the one mathematically correct answer. But especially utterances constructed in such a manner could be matter in dispute, if, in accordance to Wood, the teacher makes an argumentative discussion possible and the discursive argumentation of different meanings becomes subject of classroom interaction. Whether or not the discourse leads to consensus does not matter for fundamental learning taking place in these situations:

"For the triggering of structural learning processes, getting a consensus about the question of dissent is not even necessary. Only a method of mutual understanding of differences is required; and the more complex and non-transparent the differences are, the more radical and profound learning can be." (Miller, 2006, pp. 217–218, translated by the authors)

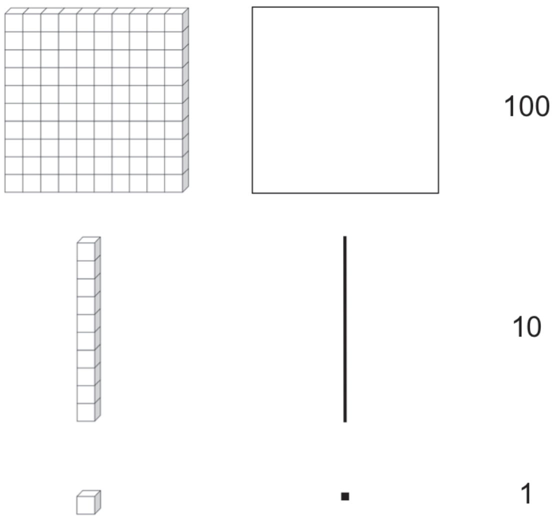

Figure 1 Iconic and symbolic representation of the Dienes-material

For example, in (German) primary schools the use of Dienes-material[1] is common practice. According to Voigt, a didactic-curricular motivated, mathematically related (teacher's) interpretation would mean that squares represent hundreds, bars represent tens and dots represent ones (Figure 1). However, possible children's productions of meaning (influenced by their daily life or by [mathematics] lessons) could be – for example – the interpretation of the squares as ones-cubes, the bars as tallies and/or the little dots as reversible tiles. These interpretations might be partly competitive, many more might be possible and – depending on the point of view – they might be matter in dispute.

1 Named after Zoltan Paul Dienes

 Another example is the ambiguity of a number-line without numbers (Figure 2). The little boxes can be filled in differently. For example, the starting point, the section, the unit and/or the scale can be varied, especially because these choices could evolve into possible crucial points in the interaction.

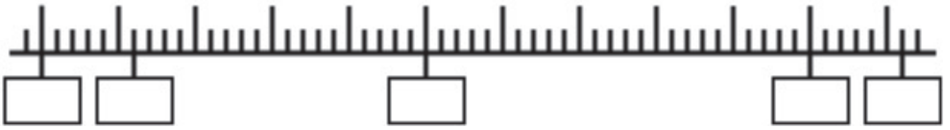

Figure 2 Number-line without numbers

3 Data Collection

The study “Probing and Evaluating Focusing Interaction Strategies in Elementary Mathematics Teaching (ProFIT)” was conducted at a primary school in year 3 and 4 in Germany. Having the theoretical background in mind, the teacher’s role in the project “ProFIT” should be to focus the mutual attention at the crucial point of a problem, to raise a question in order to hand the discussion back to the pupils and to give them the responsibility of clarifying the mathematical situation themselves (Wood, 1994). Through such a form of interaction, needs of mathematical reasoning are emphasised as a central research topic and different pupils’ interpretations have to be compared. So the main research interest is to develop and reconstruct discursive mathematical interaction as a theoretical construct of mathematics education called “focusing interaction strategies”, in which the central characteristics of negotiation processes between teacher and students are classified. Hence, the research interest is directed on two perspectives:

- How can the teacher make a need for more mathematical reasoning more accessible for pupils? And: In which way can the needs be developed in the common discourse?
- How can pupils understand a mathematical reasoning need? And: How are they able to agree on it?

Five teachers participated in the project. In several videotaped small group discussions, each of them discussed four mathematical topics with several groups of four children. The topics’ main characteristics were the openness with regard to ambiguous conventional aspects (Steinbring, 1994, see the two examples above) and operational connections in problem-based tasks (Wittmann, 1995). So the crucial points of these tasks can be the different interpretations of the mathematical signs/symbols or the different ways of solving problems. The theoretical construct is particularly elaborated by theory-based, interpretative analysis of the video data of these experimentally planned mathematical discourses with pupils.

4 Data Analyses

For the data analyses we take two theoretical points of view into consideration: the epistemological and the evolving steps of debate in a broader unit of meaning. The epistemological point of view deals with the fact that it is impossible to analyse discourse without seeing the contributions of the interlocutors in the interaction. Hence, the outcome of an interaction often becomes only visible in the particular utterances. Against this background, the data analysis starts with the epistemological analysis of the meanings of mathematical signs and symbols being interactively constructed by the pupils (Steinbring, 2005). This epistemological analysis is embedded in the analysis of a broader episode, considering the three evolving steps of dispute: Initiation – Maintenance – Breakup. Beginning with a potential point of dispute based on different interpretations; whether the interaction about the differing representations is pursued or dropped will be analysed in the next step. In the latter cases, the dispute breaks up at once. If we follow the interaction, we look more closely at the way of maintenance. This is analysed with two questions in mind:

- Is the interaction more teacher-centred, which means that the pupils’ changing and partly not conventional interpretations are taken as “mistakes”? In this case, the “correct” conventional use of signs and symbols is transmitted by the teacher and the interaction is led to consensus. Such an interaction is regulated by the teacher’s logic of interaction.
- Or does the interaction focus on the special mathematical content? In this case, the question of debate is not only realised, but substantiated and deepened. A possibly emerging consensus is rather based on the subject matter’s logic instead of the logic of interaction.

4.1 First Example: Dispute about number representations

The task “Find different representations for the number 417!” promotes classroom discourse about the previously produced solutions of the pupils, as shown in figure 3.

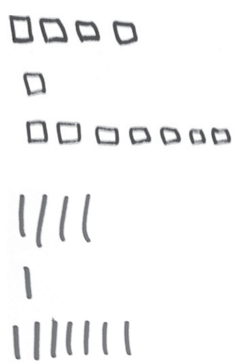

Figure 3 Pupils’ solutions for representing 417

Here, important differences to conventional representations become visible. Earlier, there was a conventional use of the Dienes-material: "little cubes" representing ones, "bars of each ten cubes" representing tens and "squares of each a hundred cubes" representing hundreds. In contrast to this interpretation, some of the elements of the Dienes-material now have been given another representing role. In figure 3, two representations of the number 417 are shown, one using the bars and the other the squares of the Dienes-material in a new way. For representing a number value, the perceptible form of squares or bars is not taken, but the value of hundreds, tens and ones depends on spatial relations between the same elements. These two representations are in the focus of discourse in the following episode.

Ben interprets the "squares" representation of 417:

B The one with the squares, there are always, um (…). On top there are four squares, that's supposed to be the four (…), the four hundred and the one is of the, seventeen the one and those seven squares are supposed to be the seven of seventeen.

T Can you show me that more precisely? I cannot see that. (…)

B Well this (*goes over the four squares in the first line with his finger*) is supposed to be the, four hundred the four squares the single square (*points with his finger at the single square in the middle*) is supposed to be (…) ten and that (*goes over the seven squares at the bottom line with his finger*) are supposed to be seven.

In his explanations, Ben exclusively refers to the representation with "squares". He attributes a numerical value to every square, varied by different spatial relations within the representation. His choice of words "is supposed to be" emphasises the interpretation character of his utterance. The teacher's intervention leads him towards pointing precisely at the representations, but not towards verbalising the arrangement any further.

Manuel carries out another interpretation:

M That are, um, but however the bars are the tens and the units (*points at the poster with the bars*).

He takes the representation with the bars into consideration and interprets the bars as "tens and units". What he understands exactly as ones and tens does not become clear. Possibly, he interprets them as tallies (ones) or as the tens-bars of the Dienes-material. By saying "are" he takes a more or less definitive perspective towards the representation, which indicates that each symbol represents something special, possibly fixed. With the German word "doch" (in English "but however") Manuel separates his own interpretation from Ben's, so that his utterance as response to Ben's has the potential for being in dispute.

Ben negates Manuel's arguments indirectly:

B That is the same as this one just with bars (*points in turns to the poster with bars, then at the representation with squares*).

What is new in Ben's contribution? He accepts the dissent and, hence, takes both representations into account. He speaks about "the same", but he does not show his understanding of "the same". The representations are not the same, but they have the same structure and the individual symbols have the same local positions. This defends Ben's argument that each of these presentations could be 417.
Frank joins in:

F Um, the ones are the, these little cubes. And the big, and the big squares are the hundreds and the bars are the tens.

Frank's utterance shows a different understanding from Ben's and Manuel's. He becomes a bit more concrete than Manuel and directly labels the conventional representations of the Dienes-material. His utterance is not necessarily linked to the representations in front, but he highlights special features exclusively connected to the single objects of the Dienes-material (size, form). Frank does not refer at all to the spatial relations of the elements in figure 3.

Until now, a lot has happened during the interaction. Ben starts with a relational interpretation which is limited to the representation of squares. However, Manuel interprets the materials by directly allocating numerical values to the concrete objects. Furthermore, he raises a new focus: the representation with bars. Ben accepts the dissent by expanding his spatial relational interpretation of the squares to the other representation using the bars. Frank also interprets the material and explicitly designates the concrete characteristics of certain symbols. Taking the dispute more closely into consideration, it is maintained by the different utterances. The teacher does not intervene by aiming at a conventional interpretation. Instead, the pupils have an opportunity to discuss a mathematical content. Following the interaction, other pupils join in with their points of views such as "If you add all tens together, then it equals one hundred and twenty. And here those hundreds equal one thousand two hundred (*points at the representation with bars first, then at the other representation*)" and „Um, the top squares are always the hundreds, beneath there are bars, that is the tens, and beneath the ones, so four hundred and seventeen." Both, the empirically-concrete and the relational interpretations become more and more specific during the interaction.

After John has explained "actually it doesn't matter, if those are squares or bars. […] You can also use what you haven't learned yet. Form something from that", the discussion breaks off. For John, the concrete features do not make any difference. Kevin enhances this point of view by exemplifying this comparison:

K Then you can also, if … I think I know what John means. Then you could also make a poster, where, those four (*points at the four bars at the top*) are dots and then also here (*points at the single bar*) one dot for the tens and, and then here (*points at the seven bars at the bottom*) seven dots. You could also do that actually.

He structurally compares the representations with a not yet produced new representation consisting of dots arranged in the same way. The flexibility of his point of view becomes visible in the confirming statement: “You could also do that, actually,” that John formulates subjunctively as a conclusion.

Discussion Episode 1

In this episode, the disputed point does not get solved. Instead, the characteristics of number representation are discussed by using different points of reference, based on distinguishing features on the one hand, and on the relationship between particular elements of the same representation or in comparison to other representations on the other.

Both points of view can be legitimised in a particular manner, are permitted depending on the conditions and are substantiated respectively in the interaction. According to this, by using the Dienes-material the focus is put on the special external characteristics in the first instance. The relations of the Dienes-material are given by combining ten ones to one ten or ten tens to one hundred etc. No matter in which spatial position each symbol is put, the numerical value stays the same. The Dienes-material together with their ideal visual pictures might offer direct relations between the positions of ones, tens and hundreds by counting ten little cubes in a ten bar and ten ten-bars in a hundred square. Then again, as special kind of an additive number system this material is less flexible – it does not emphasise the decimal number *structure*. The system of positions yields advantages; however, most of them are not directly visible, they must be read into the representation. In the position of hundreds, the digit 4 of the number 417 means something completely different than if it was in the position of the tens. This issue is exactly the cause of the mathematical dissent in the course of this episode.

The teacher allows the negotiation of the dissent without breaking it off at once. She intervenes only in very few situations. In this episode, she does not need to, because the pupils’ debate arises on its own – with the focus on the mathematical content. Moreover, in this episode the teacher’s interventions are directly connected to the mathematical content and targeted at the pupils’ utterances, as for instance, in the following situation:

T I think, I know what Frank means. When I turn it like this (*turns the poster at the bottom through 180°*) look, then I can I have a square for each bar. (…) (*repeats pointing four times from the last bar at the bottom* (*in the bar-representation of 417*) *to the last square at the bottom* (*in the square-representation of 417*) (*fig.3*)) Is that, is that what you meant?

The analysis of this short episode shows no teacher-centred interaction between the participants in which the students would have to follow the implicit goals and intentions of the teacher.

4.2 Second Example: Dispute about finding a number in a number-line

In this episode Julian (Ju), Sascha (S), Frank, Janina (Ja) and another teacher (T) than in Episode 1 discuss a number-line with a scale in steps of 20 (produced by Janina in the previous lesson):

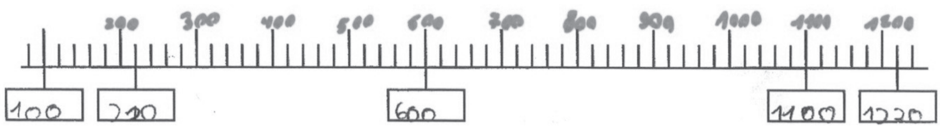

Figure 4 Janina’s number line with inserted numbers

For a better orientation, the pupils write the hundreds at the long bars (Figure 4) before the episode starts.

T What would happen if we looked for the number three-hundred-fifty? Did you write it onto the number-line or look for it?
S It can’t be.
Ju But three-hundred-twenty, thee-hundred and forty, (*is pointing one after the other to the two short bars after the 300-bar*) there in between (*is pointing between the two short bars between the 300- and 400-bar*). There in between.
Ja Yes.

The teacher asks whether or not the number 350 can be found on this number-line. With this question, a problem of the epistemological status of mathematical knowledge is touched upon. From an epistemological point of view (Steinbring, 2005), the teacher introduces a new sign/symbol that needs to be explained by relating this sign to a known explaining reference context. Sascha and Julian interpret the number-line in different ways as explaining reference contexts. Sascha principally argues that numbers can only exist on the number-line when having a pre-given scaling bar to place the number at. In contrast, Julian’s view also allows numbers on the number-line to be placed between scaling bars. In this way, the teacher’s question evolves as a question of debate.

Sascha denies the teacher’s question, at first without any further explanation. In contrast, Julian tries to support his statement by pointing at a location where the number 350 could be exactly and by explaining the conceptual relations of the number-line in a geometrical-relational way. Janina confirms this. In the following interaction, the teacher intervenes by asking Sascha: “Why doesn’t that work?” It is unclear if the disputed point had continued without this intervention. Sascha explains his interpretation more precisely:

114 S Um (…) because (*takes Julian's finger off the number-line and points at the number-line himself*) […] if you um took it in steps of ten because then there would be half that is three-f … three-hundred and fifty but if you had bigger […], because then there (*moves his finger under the number-line from the long bar at the second box to the middle*) would be for example three-hundred (*points at the long bar with the number 300*) and then there four-hundred (*points first at the long bar with the number 400, then at the long bar with the number 500*) at the long one exactly in the middle (..) and because here, um, always ten, twenty, thirty, forty and then fifty (*one after the other, he points at the short bars behind the long bar with the number 300*) then you would already have three-hundred and fifty and with twenties you cannot find it (*1 sec*) twenty forty sixty eighty (*one after the other, he points at the short bars after the long bar with the number 300*).

Sascha does not establish relationships between different number-lines but only refers to the number-line at hand, which he and his group have labelled with numbers (Figure 4), especially at the interval between the label 300 and the label 500. By changing the number 500 into 400 and the steps of twenties into steps of tens hypothetically, he affirms that – if the number-line underlies a decade-structure – the number 350 could be found or marked because a concrete scaling bar exists there to place that number at. He supports this claim in an empirical-descriptive way by referring to the number-line, pointing at it and counting aloud while changing the units. The statement implies a norm which he does not put into words, but it seems to be plausible because of his pointing gestures: You can only find a number if a scaling bar at the number-line exists. This is supported by the use of "always ten" as unit, as he is exemplarily marking the steps of tens up to 50 and by the derivation of the number 350 at the long bar ("at the long one exactly in the middle").

Sascha specifies and rephrases his claim "It can't be" in the last part of his statement by saying "… with twenties you cannot find it". With this, he refers in a reifying way to the actually perceptible objects on the number-line, which he is using as counting-objects. Concerning this statement it remains unclear, if the numerals "twenty forty sixty eighty" refer to a more geometrical interpretation or to counting in steps of twenty, but therefore Sascha indirectly refers to the norm of steps of twenties, which are jointly accepted by the interlocutors, accompanied by the pointing gesture with his finger from bar to bar.

Next, the teacher addresses Julian. Julian reacts to the teacher's demand with several contributions, which are partly interrupted by Sascha:

T Julian, think, I want to find the number three-hundred and fifty on this number-line, can I do that?

Ju I, so when I [count] in steps of ten […] (*turns the number-line to himself*). If it was in steps of ten, then it would be possible but if one [counted] in twenties (*shakes the head*), then it would not be possible.

T So at, um, with these steps (*points underneath the number-line between 300 and 400*) at this number-line (*taps with the hand on the whole number-line*) I cannot find that?
S You can’t (*shakes his head*). No.
J But, in fact, if you [looked] in the middle (*points to the two small bars in the middle of the long bars with the numbers 300 and 400*) but that (*shakes the head*)…

Further, Julian justifies his interpretation of the signs/symbols. At the beginning, he reads the number 350 into the number-line at hand (figure 4). For Julian, the number 350 can be found between the two middle bars between the long bar with the label 300 and the long bar with the label 400. This interpretation can be interpreted in such a way that all numbers can be found on the horizontal line at which the scaling bars are marked. The bars merely help to find them. During the discussion, he changes his view and agrees for a moment with Sascha that the number 350 cannot be found. He confirms the correctness of Sascha’s claim by paraphrasing him. At the end of the scene, Julian tries again to reinforce his claim: “But, in fact, if you [looked] in the middle (*points to the two small bars in the middle of the long bars with the numbers 300 and 400*)”. His geometrical-relational interpretation permits to construct a number on the number-line even if there is no bar for it. But as he cannot explain this interpretation, he rejects it. It seems that his own arguments are neither sufficient nor convincing for himself.

The dispute stops with Julian shaking his head. To sum up the already analysed processes, Sascha’s claim that the number 350 cannot be found in the given number-line and Julian’s counter-argument “in the middle” remain opposed to each other. The question whether or not the number 350 “exists” on the number-line presented (Figure 4), in this case leads to a cooperative argumentation (Klein, 1980). Indeed, Sascha and Julian express their contradiction spontaneously at first. They take their points of view but also try to modulate their different positions to a mutually accepted statement. Looking back at this whole episode, one gets the impression that, with the help of the teacher, Julian is trying to find arguments for or against the teacher’s question if one can find the number 350 on this number-line; so Julian is not only looking for arguments supporting his own position.

Both positions cannot be refuted, but their justification background is different. It remains unclear whether Julian’s argument stays permissible or not, even if it cannot be transferred to be collectively valid, so to speak by the “exclusion principle”. There is no further reflection of Sascha’s statement.

Discussion Episode 2

Also in this second episode, the disputed point is not solved. Instead, some characteristics of number-lines are discussed by using different points of reference in the pupils’ interpretations: Hypothetical conceptual variation in a number-line, the reifying empirical-concrete interpretation which is based on the bars, and the geometrical-relational relations between particular elements of a filled in number-line. All points of view can be legitimised in a particular manner; they are permitted depending on the conditions and they are respectively substantiated in the interaction.

Looking at the teacher's role during the interaction process and how her role refers to the learning of mathematics, it remains unclear if the discussion had continued without the teacher's intervention or if the interlocutors accepted Sascha's utterance as sufficient justification for his claim. The teacher constantly looks at the pupils' interpretations. Furthermore, she takes care of mathematical reasoning in the pupils' interpretation of the presented mathematical signs and symbols. Her interventions initiate a more profound review of the pupils' contributions. The claims, which are at first not justified, become explanations linked to mathematics. Moreover, in the moment in which the number 350 cannot be interpreted in the number-line, the teacher does not take over a transmitting role. The episode shows that a disputed point arising in a mathematical discussion can remain contradicting for students, whereas the teacher, because of her expert knowledge, can relate different interpretations to each other.

5 Summary

In summary, the interpretations show that it is possible to initiate and discuss mathematical crucial points with children at the elementary school age, but it is challenging to focus the attention on special aspects of the interpretation. By giving pupils the possibility of debating about mathematics in the described way, many opportunities for fundamental learning arise, like the exemplary episodes have shown. Both episodes have offered different ways of how students can cope with number representations. The first episode dealt with differences and connections between additive and positional number systems by using and changing the role of known material like the Dienes-material. The second episode dealt with reading numbers into number-lines by varying the unit measure. These ways of working with different sorts of representations offer productive starting points for mathematical debates between teacher and pupils about substantive arithmetical representations and meanings.

With regard to the two research perspectives as expressed in the research questions in the paragraph "data collection" the following first attempts for answers to these research questions can be made: Having a closer look at the teachers' role, they intervene differently during the episodes. Firstly, in episode 1 the teacher *reflects* the importance of the pupil's utterance by asking for a more extensive explanation: "Can you show me that more precisely?" By dealing more profoundly with the statement of the pupil, the teacher's intervention becomes a stimulating effect in the course of the discussion. Secondly, later in the same episode, she passes the interactive utterance back to the students by *rephrasing* Frank's argument. Also in ordinary classroom teaching, in which the teacher has more a role of conveying knowledge, this behaviour is used. The teacher picks up a pupil's contribution and restates it, rephrasing it, formulating it more precisely and introducing more mathematical notions. In this episode, the difference to a classroom setting of conveying mathematical knowledge is the way of dealing with the teacher's contribution: the pupils understand the teacher's intervention as invitation to continue their mathematical negotiation and, by that, to sharpen their understanding and progress their mathematical argumentation skills. The pupils do not try to search in the teacher's reactions for some implicit signals indicating the "correct" knowledge the teacher seems to have in mind.

In the second episode the teacher obtains a different social role. She is coming up with the question: "What would happen if we all would look for the number three-hundred-fifty?" At once, the pupils give two contrary answers, a disputed point evolves. Compared with the first episode, where the disputed point evolves by the pupils' contribution, here the teacher starts the new negotiation. Thus, the teacher *initiates* the discussion. Later in this episode, the teacher intervenes once more. This time in a *provoking* way: "[…] with these steps at this number-line I cannot find that?" Indeed, the teacher provokes once again in the discussion with regard to the supposedly agreed consensus and she reveals an opponent point of view: "No, you can't." and "But, in fact, if you looked in the middle […]" (in this case this is not resumed any more).

All four types of teachers' intervention (*reflecting, rephrasing*, *initiating* and *provoking*) can serve as productive catalysts of debate, as far as the participants of the communication understand and use them in the intended way. Thus, the teacher's role is inseparably linked to the manner in which the pupils understand the interaction and how they react themselves. In the presented episodes, the teachers' interventions lead to promote their joint mathematical content-oriented discussion. In this way, such discussions about disputed points might become crucial instances for developing a deeper structural understanding in elementary mathematical topics. It is the goal of the research project *ProFIT* to work out more exactly what and where the crucial points are and the theoretical construct *focusing interaction strategies* to be developed for characterising these crucial points.

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Mathematics in Perception of Pupils and Teachers[1]

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Abstract: The presented study shows the possibilities of increasing the teacher's professional competencies using focused self-reflection on the basis of confrontation of his or her own ideas, expectations and observations with data obtained mostly from the statements of pupils. The study is based on the results of research carried out in the Czech Republic (sample: 3108 pupils from 25 elementary schools and 179 teachers of these pupils). The attitudes towards mathematics from the point of view of pupils were analyzed, as well as the teachers' perceptions of pupils' attitudes. The monitored variables are: popularity, difficulty, and importance of the subject, self-reflection of one's own talent for the subject, how interesting pupils find the subject, and their motivation and diligence in mathematics, complemented by the pupils grades and teachers' evaluation of each pupil's performance in mathematics. The study also provides examples of the trend of German pedagogical-psychological research and shows the parallels and possible directions of comparative research in the given area.

Keywords: pedagogical psychology, motivation, mathematics, pupils perceptions and attitudes to subjects, professional competencies of teacher, diagnostic competencies of teacher, effectiveness of instruction, criteria of effective teaching

1 Introduction

The objective of every educational system is to increase the effectiveness of teaching of all subjects. This is possible only on the basis of (a) accurately defining the appropriate criteria to ensure effective teaching, (b) determining the key factors influencing effective teaching (including the power of each factor and their effects on each other), (c) systematic appreciation of the changes in effectiveness depending on changes to the relevant factors.

Comparison of the educational system with other systems then allows for voluntary positive changes in the effectiveness of teaching, with the assumption of unified criteria for measuring the effectiveness of teaching.

The main focus of our research is the teacher and his or her professional competencies (Spilková, 2008). In our study, using a pedagogical-psychological approach, we will focus on the possibilities of increasing the teacher's competencies using controlled self-reflection by confronting one's own images, expectations and observations with the data gained mostly from pupils' own statements (Hrabal & Pavelková,

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2010). Another objective of our study is to present the current trends in German pedagogical-psychological research, and point out the parallels and possibilities for comparative research in this area.

2 Possibilities of Using the Pupils' Statements in Describing the Subjects of Instruction

The pupils' statements about a subject of instruction could be used for comparison of the character of chosen subjects and their differences in observed countries, based on the statistical processing of large samples. They could be also used to determine the efficiency of instruction of individual teachers in terms of creating positive attitudes towards the given subject.

Using a sample of Czech pupils from selected secondary schools and the semantic differential method, R. Pöschl (2005) identified differences in the importance assigned by pupils to mathematics and physics. The results of his research showed that the pupils in his survey associated the terms 'mathematics' and 'physics' with the terms 'school', 'theory', 'duty', 'formula', and 'truth'. By contrast, the pupils did not connect the terms 'mathematics', and 'physics' with terms such as 'science', 'nature', 'love', 'life', or 'future'. According to Pöschl's study, pupils find physics to be 'distant', 'boring', 'ugly', and 'complicated'. The pupils considered mathematics less distant and more useful. However, the point of view of those pupils that 'like it' is different; physics is, for these pupils, much more 'useful', 'varied', 'beautiful', 'active', 'entertaining', 'young' and less 'complicated'.

In a German study, the authors surveyed pupils' impressions of the importance of individual subjects to everyday life. The questionnaire they created included among others, questions about the difficulty of subjects and their significance. The results show that, similarly with the Czech Republic, German pupils consider mathematics to be difficult but significant. However, the importance to the everyday life is quite low compared to the German language (Haag & Götz, 2012).

The research of other German authors Kessels and Hannover (2006) demonstrated the importance of investigating the pupils' impressions of individual subjects. They discovered that a pupil's impression of a subject was reflected not only in his/her approach to the given subject, but it could also strongly influence the pupil's own self-conception. Their research was aimed at surveying German pupils' impressions of natural sciences and comparing and contrasting these with the impressions that pupils who preferred these subjects associated with them. The authors show on the example of physics that some subjects have a very distinctive image, and that pupils have the tendency to regulate their own identities by focusing their interests in subjects according to this image. As the authors of the study state, the pupils thus implicitly affect their own self-image.

3 Pupils' Approach to School Subjects and Their Self-Assessment in the Area of Qualifications for Learning

3.1 Example of the research orientation in Germany

H. Ditton (2002) highlighted the potential of using pupils' statements to increase the effectiveness of teaching. On the basis of the analysis of many research works, Ditton came to the conclusion that how pupils perceive the teaching is an important factor when determining the professional qualities of the teacher. He carried out large scale research in which he used a sample of 4316 pupils in 186 ninth grade classes, who assessed their mathematics teachers both on their behaviour during the instruction and the characteristics of the instruction. Ditton found that a positive attitude towards mathematics teachers statistically significantly correlated with the values of importance the pupils give mathematics ($r = 0.31$), their fear of mathematics ($r = 0.32$), how fair they consider the marks from oral examination ($r = 0.46$), how interesting they find the instruction ($r = 0.64$), how they assess the teacher's diagnostic competencies ($r = 0.73$) and how clear they find the instruction ($r = 0.65$). On a partial sample of 172 pupils, the author further investigated how the assessment of pupils corresponds to the expectations of the teacher. The correlation between how interesting and how well structured the teaching was, was between $r = 0.39$ and $r = 0.44$, and the correlation between the effectiveness of the teacher's classroom management and the positivity of the teacher–pupil relationship was $r = 0.48$. The author concludes: "How the pupils perceive instruction could be an important base for improving the quality of the instruction, if data from such surveys aren't used only for research purposes, but also made available to the teachers." As part of the study, the results were given to each of the teachers and they could compare their results with the results of the whole sample. The feedback showed that many of the teachers used such obtained information to improve the quality of their teaching. Our concept of the self-diagnostic possibilities of the teacher (Hrabal & Pavelková, 2010) (see also part 5 of this study) is of a similar vein, but it is further developed in the area of diagnostic methods.

3.2 Our research study

Research sample

Our research was conducted between 2005 and 2008 in two distinct phases. Pupils at the lower secondary school (grades 6 to 9) participated in both phases. Only the pupils were surveyed during the first phase in 2005 and 2006. During the second phase in 2007 and 2008 their teachers also participated. A total of 2071 pupils from 101 classes in 18 schools were investigated in the first phase, with an additional 1037 pupils, and 179 teachers, from 50 classes in 7 schools in the second phase. The schools were carefully selected to ensure that they fairly represented the full spectrum of schools nationwide in the Czech Republic. We therefore had a total sample size of 3108 pupils from 151 elementary school classes.

Table 1 Numbers of monitored pupils

	1st phase			2nd phase			Total		
Year	Boys	Girls	Total	Boys	Girls	Total	Boys	Girls	Total
6	282	259	541	102	80	182	384	339	723
7	280	250	530	130	104	234	410	354	764
8	262	242	504	152	131	283	414	373	787
9	282	214	496	169	169	338	451	383	834
Total	1106	965	2071	553	484	1037	1659	1449	3108

The research method and chosen variables

The research method used on the pupils in our study was *Questionnaire about attitudes to subjects* (Hrabal & Pavelková 2010). The questionnaire included the questions regarding the character of the subject of instruction, one's own preconditions for success in the subject, and information about the pupils' marks on their last school report. The questionnaire used always a 5-level scale:

- *subject popularity* (1 – very popular subject … 5 – very unpopular subject);
- *subject difficulty* (1 – very difficult subject … 5 – very easy subject);
- *subject importance* (1 – very important subject … 5 – not an important subject);
- *mark on the latest school report;*
- *talent for the subject* (1 – very talented … 5 – not talented);
- *motivation in the subject* (1 – very motivated … 5 – unmotivated);
- *diligence in the subject* – (1 – very diligent … 5 not working/lazy).

Characteristics of the monitored variables

- Subject popularity – emotional experience of the subject (both as a precondition and as a result of the motivation to learn).
- Perceived difficulty of the subject – and the relation to many motivational processes (feeling of powerlessness, self-image, success or failure, etc.).
- Subject importance – ascribed subjective value of studying the subject. Source of motivation – internalisation of the social representation of the subject via its application in society.
- Self-perception of one's own talent – the competence component affecting motivation to learn.
- Diligence – realised motivation in lessons and home preparation.

Teachers of individual subjects assessed their pupils using the same characteristics as the pupils used to grade themselves. They also assessed the performance of the pupils on the 5-level scale (1 – very good performance … 5 – very poor performance). The teachers therefore tried to assess their pupils' attitudes towards individual school subjects.

These variables were measured for the following school subjects in the given year: Czech language (Čj), mathematics (M), English language (Aj), German language (Nj), physics (F), chemistry (Ch), biology (Př), geography (Z), history (D), citizenship (Ov), Family education (Rv), arts (Vv), music (Hv), sports (Tv), work education (Pv), informatics (I). Abbreviations in parentheses are taken from the Czech names of the subjects. A summary of results follows. We have specifically selected the data that demonstrate the implicit concept of teaching mathematics. Mutual relationships between the variables were identified on the basis of correlation coefficients with the significance border $p < 0.05$.

Results of the study – relationship between popularity, difficulty and importance of the subject

As set out above, and detailed further in our research (Hrabal & Pavelková, 2010), the popularity, difficulty and importance of the subject to the pupils are all important components in their motivation to learn, and by influencing one or more of these components it is possible to increase the efficiency of the instruction. They are also important indicators characterising the subject and feedback given by the pupils, which can then be used by the teachers for their self-reflection.

Relationship between popularity and difficulty. Generally, it is possible to say that the more difficult a subject is, the less popular it is with the pupils. Typically, if a subject is perceived to be particularly difficult, it tends to be particularly unpopular, and vice versa. If we consider extreme groups from this point of view (a popular and very popular subject versus an unpopular and very unpopular subject), there is a strong relationship between popularity and difficulty. However, we have found that for another group of pupils this tendency does not exist and, indeed, the contrary appears to be true. Up to a quarter of pupils that liked a given subject a lot, considered it to be difficult or very difficult. This phenomenon was found to be present in subjects such as the Czech language and mathematics. Forty percent of those who liked mathematics, considered it to be easy or very easy, but 25% of pupils that considered it to be difficult or very difficult, also liked it very much. Mathematics and the Czech language generally belong to the rather unpopular and difficult group of subjects.

Relationship between popularity and importance of the subject. In terms of the relationship between popularity and importance of the subjects, we again noticed positive, although less pronounced trends; typically the more popular a subject is among the pupils, the more important they find it. Conversely, the more important a given subject is for the pupils, the more they like it. (Causal relationships between popularity and importance were not a subject of our study.) What is significant is that the relationship between popularity and importance exists, is of moderate strength, and is true for both the subjects generally considered to be popular (arts, music, sports, citizenship, family education, etc.) and for the subjects considered to be less popular (mathematics, physics, etc.).

Relationship between difficulty and importance of the subject. The relationship between difficulty and importance was found to be less pronounced and less stable than either of those described above. For mathematics and most other subjects, there was only a relatively weak mutual connection between these factors. For Czech language, arts, music, etc., there was no discernible relationship between difficulty and importance shown in the results.

Relationship between popularity of the subject and mark. Teachers frequently believe that if a pupil is successful in a subject, he/she will like it. But the results of our study do not entirely support this claim. There was only a weak correlation between the popularity and results for most subjects; in particular, there was only a medium-strength connection between mathematics, geography, history, English, and German language.

Relationship between difficulty of the subject and mark. A medium-strength relationship was established between perceived difficulty and results in the subjects of mathematics, Czech language, geography, history, English language, German language and citizenship. For the rest of the monitored subjects, the research found only a weak relationship between these factors. Thus, it is not possible to state with any degree of certainty that pupils with bad results in a given subject must therefore consider it to be particularly difficult, or vice versa, that pupils with good results consider the subject to be easy.

Relationship between importance of the subject and mark. A weak relationship was found to exist for most of the subjects between the importance of the subject and marks.

Results of the study in terms of homogeneity and heterogeneity of pupils' attitudes towards subjects

The level of homogeneity or heterogeneity in pupils' attitudes towards certain subjects may be a better indicator of the real level of agreement or disagreement in the attitudes, and thus may more clearly demonstrate the social standing of individual subjects. From the point of view of the teachers, there is a great difference between teaching a subject where the attitudes are similar (for example, where almost all pupils like it or almost all consider it difficult) and a subject where the attitudes of pupils are differentiated. It is important to note the results with regard to the levels of homogeneity with a degree of caution given that they could, of course, be influenced by a number of subliminal and/or external factors. These could include the difficulty pupils may have in reflecting on or differentiating their particular attitudes towards a subject, and/or the product of their indecisiveness, lack of clarity of their opinions, and/or stronger tendency to choose middle values.

To assess the homogeneity of individual attitudinal characteristics, we first sorted them according to the standard variation. In the second stage, we divided the 112 characteristics (16 subject × 7 characteristics) into quartiles (Table 2). The subjects in the first quartile have a high level of homogeneity, whereas the subjects in the fourth quartile have a low level of homogeneity or a high level of heterogeneity of pupils' attitudes towards them.

Table 2 Homogeneity and heterogeneity of pupils' attitudes towards subjects

Characteristic	Subject	Subject Characteristics
	High level of homogeneity	
Marks	Vv, Hv, Ov, Tv, I	very good
	Př, Ch, Z	good
	Čj	bad
Importance	Aj, M, Čj	high
Talent	Tv, Rv, I	good
	Př, Ov, Z	middle
	Čj	low
Popularity	Tv, I	high
	Čj	low
	Low level of homogeneity	
Interest	M, F, D, Ch, Vv, Hv, Ov, Tv	nothing typical
Diligence	all subjects	nothing typical
Motivation	all subjects	nothing typical
Popularity	D, Ch, M	nothing typical

The abbreviations are taken from the Czech names of the subjects as follows: Vv – arts, Hv – music, Ov – citizenship, Tv – sports, I – informatics, Př – biology, Ch – chemistry, Z – geography, Čj – Czech language, Aj – English language, M – mathematics, Rv – family education.

Mathematics – summary. Pupils' attitudes towards mathematics are quite differentiated; the pupils agree on the high level of importance of mathematics, but their attitudes differ in the popularity and interest of mathematics as a subject, as well as in their motivation and diligence in mathematics.

Results of the study in the area of pupils' approaches to mathematics

Mathematics as a subject is perceived as one of the least popular – it was the third most unpopular subject, ahead of only German language and Physics. Mathematics was also viewed as the most difficult subject of all those investigated, and at the same time, the pupils perceived it as a very important subject (third place behind English language and Czech language). The average results in mathematics were the worst of all the monitored subjects, with pupils considering themselves less talented at mathematics (third worst of the subjects, in front of only Czech language and physics). Pupils' motivation was found to be only average – in comparison with other subjects, mathematics came in fourth place behind informatics, English and sports). With regard to diligence, pupils considered themselves to be only averagely hard-working in mathematics (sixth least diligent subject). More detailed information about all the subjects in the lower secondary schools are summarized in the book by Hrabal and Pavelková (2010).

Gender differences

We also looked at the differences between boys and girls in their respective attitudes towards mathematics. The results are shown in Table 3. Due to the large sample size, statistical significance could be also found by small differences. On average, the difference between boys and girls was not greater than ±0.20, approximately a quarter of a grade. The differences between boys and girls in their attitudes towards mathematics were not therefore generally statistically significant. The greatest differences were found to exist in the perceived talent that boys and girls consider themselves as having for mathematics, where the boys felt more talented than the girls (difference 0.27), and in the motivation that each has to learn mathematics – where once again the boys felt more motivated (difference 0.23). It is interesting that there was no significant difference in diligence (diligence as real motivation). Other differences included that mathematics was more popular with boys than girls (difference of averages 0.13), that boys also consider it easier (difference 0.18), and that boys as a whole have better marks (difference 0.19). Differences between the approach of boys and girls to other subjects can be found in the study by Pavelková (2005).

Table 3 Differences in approach to mathematics between boys and girls

		N	Average	Difference in averages	Importance
Popularity	Boys	1658	2.84	−0.13	**
	Girls	1448	2.97		
Difficultness	Boys	1659	2.73	0.18	**
	Girls	1446	2.55		
Importance	Boys	1658	1.75	−0.08	*
	Girls	1447	1.83		
Mark	Boys	1646	2.55	0.19	**
	Girls	1441	2.36		
Talent	Boys	852	2.71	−0.27	**
	Girls	760	2.98		
Motivation	Boys	851	2.47	−0.23	**
	Girls	762	2.70		
Diligence	Boys	850	2.71	0.09	
	Girls	759	2.62		

* = $p < 0.05$, ** = $p < 0.01$

4 Diagnostic Competencies of Teachers in Respect of Pupils' Approaches to Mathematics

4.1 Research of diagnostic competencies of teachers in Germany

Research activity aimed at the diagnostic competencies of teachers in Germany was provoked by disappointment in the results in the PISA study (2002), and also by the conclusions of the Conference of Ministers of Culture of the German States (2004) that stated low diagnostic competencies of the teachers as one of the reasons for the PISA study results. In this regard we make reference to two studies on this theme. The first study (Praetorius et al., 2011) looked at precisely how teachers know to evaluate the self-image of pupils in the area of competencies in individual subjects, i.e. what is their diagnostic competence in this area. It compared the *level* – average value of the self-image and teacher's estimation, *differentiation* – tendency of the teachers to overestimate or underestimate the self-image of pupils and *ranking* – the difference in the order of pupils in class created on the basis of teacher's estimations and the pupils' self-assessment. The research was based on a sample of 663 pupils of the first year of elementary school, and 37 of their teachers from 20 schools. The results show that there is no difference between subjects and length of experience of the teachers. Only differences between pupils and teachers were found in the area of ranking.

In another study (Karing, 2009) aimed at the diagnostic competencies of elementary school and grammar school teachers, the accuracy of the estimation of performance and subject motivation of pupils was investigated. The research was conducted in Germany on the sample of 1984 pupils from the 4th year of elementary school with142 of their teachers, and 914 pupils of 5th year of grammar school with 111 of their teachers. Performance tests were made in the areas of vocabulary, understanding of texts and calculations. Also surveyed were interests of pupils in German language and mathematics. The results showed that the teachers at the elementary school estimated the pupils' performances more accurately than the grammar school teachers. In the evaluation of interest in mathematics, both groups of teachers had similar results.

4.2 Results of our study for teachers of mathematics

As we have already shown, pupils have typical attitudes to individual school subjects. The subjects in pupils' self perception differ in all monitored indicators. We will now focus on the teachers' perceptions of pupils' attitudes. In this sense, we do not consider the comparison of teacher's impressions of pupils' self-image with the pupils' self-image. We for now only present a general comparison of the views of the pupils and the views of their teachers. Table 4 shows the results for mathematics only, for the results in other subjects see (Hrabal & Pavelková, 2010; Pavelková & Škaloudová, 2008).

Table 4 Teacher's perception of pupils' attitudes towards mathematics

	Pupils (6th to 9th year)		Teachers (6th to 9th year)
Popularity	2.7; 3; 2.9; 3.0		2.2; 2.7; 2.9; 2.9
Difficultness	3.0; 2.7; 2.5; 2.4		3.2; 2.7. 2.7. 2.6
Importance	1.8; 1.9; 1.7; 1.7	Performance	2.1; 2.6; 2.5; 2.2
Mark	2.2; 2.5; 2.6; 2.5		2.5; 2.8; 2.8; 2.7
Talent	2.7; 2.9; 2.9; 2.9		2.6; 2.8; 2.8; 2.8
Motivation	2.6; 2.6; 2.5; 2.6		2.4; 2.8; 2.7; 2.5
Diligence	2.5; 2.8; 2.6; 2.7		2.4; 2.7; 2.7; 2.6

The development of pupils' attitudes towards mathematics could be summarised by saying that the popularity of mathematics decreases after the sixth year, coinciding with worsening marks after this year. Meanwhile the perceived level of difficulty of mathematics is constantly rising and the importance of mathematics is slowly rising. Talent for mathematics is perceived as the highest by pupils in the 6th year, after which it falls before plateauing and remaining largely constant. Motivation to learn in mathematics doesn't change although diligence falls in the 7th year.

In summary of the teachers' views of pupils' attitudes, we can see similar trends as with the results of the pupils' self-evaluation. There are some notable differences, for example in the overestimation of pupils' enjoyment of mathematics in the sixth year, underestimation of the difficulty of the subject (pupils consider mathematics to be more difficult) and particularly in the underestimation by teachers of the level of importance of mathematics that is perceived by the pupils.

Interestingly, the research found that when comparing the results of the teachers' estimations of pupils' performances – the teachers consistently tended to assess the performance as worse than the marks they themselves gave the pupils. They slightly overestimated the motivation of the pupils and underestimated their diligence.

5 Possibilities of Teacher's Self-Diagnosis on the Basis of the Pupils Classification Analysis

5.1 Mark and teachers' estimation of the performance in mathematics

As the above mentioned results show, the marks given by the teachers are not in clear accord with their evaluation of the pupils' performance, contrary to what one might expect. The marks were on average better than the estimated performance with the individual differences shown in Table 5.

Table 5 Frequency table: results in mathematics and estimated performance in mathematics

		Estimated performance of the pupil					
		1	**2**	**3**	**4**	**5**	Total
Mark	**1**	54	34	5	1	0	94
	2	21	111	69	7	0	208
	3	4	31	124	38	1	198
	4	1	4	20	53	32	110
	5	0	0	0	1	5	6
	Total	80	180	218	100	38	616

Marks and teachers' estimation of the pupils' performance in mathematics:
- Performance assessment corresponds to the mark = 47%.
- The performance is assessed better than the mark = 33%.
- The performance is assessed worse than the mark = 20%.

The difference between the marks and the estimated performance can be explained by the fact that the marks contain both performance and non-performance components. The non-performance components include the motivational potential of the marks, but may also incorporate the relationship between teacher and pupil, a degree of sympathy or antipathy towards the pupil. The teachers may also be under a degree of external or internal pressure in certain subjects to award good marks.

5.2 Diligence and talent as perceived by both by pupils and mathematics teachers

Share of talent and diligence in mathematics

In our research we were also interested in further comparative analysis to consider the relationship between the diligence and talent of the pupils in the opinion of their teachers, compared with the diligence and talent of the pupils in their own opinion. Once again we only used data related to mathematics in the following comparison.

View of teachers:
The level of diligence is the same as the level of talent = 53%.
The level of diligence is higher than the level of talent = 16%.
The level of diligence is lower than the level of talent = 31%.

In more than half of the cases, the teachers of mathematics ascribe the same level of diligence and talent to the pupils. This opens the question whether the teachers are able to differentiate between these two indicators. Low differentiation suggests lower diagnostic competencies of the teacher and could lead to little sensitivity to the motivation of pupils. Our research also highlights that teachers of mathematics consider almost one third of their pupils to have more talent than diligence, a view not shared, for example, by teachers of music who typically have the opposite point of view.

View of pupils
The level of diligence is the same as the level of talent = 47%.
The level of diligence is higher than the level of talent = 19%.
The level of diligence is lower than the level of talent = 34%.

Also from the point of view of pupils are the diligence and talent in many cases balanced, but if we compare these data with the perception of teacher, we could see that they distinguish more between diligence and talent.

Marks and performance in relation to diligence and talent, as estimated by the teachers of mathematics

If we compare the teachers' perception of the pupils' performance and the marks which they awarded, with how they assessed the diligence and talent of the pupils, we obtain valuable information about how performance, marks, diligence and talent correspond with each other. The results of our research in this regard are as follows:

The performance corresponds to the assessment of both diligence and talent = 45%.
The performance corresponds to the assessment of diligence and not talent= 15%.
The performance corresponds to the assessment of talent and not diligence = 27%.
The performance doesn't correspond to the assessment of either diligence or talent = 14%.

The mark corresponds to the assessment of both diligence and talent = 32%.
The mark corresponds to the assessment of diligence and not talent = 17%.
The mark corresponds to the assessment of talent and not diligence = 22%.
The mark doesn't correspond to the assessment of either diligence or talent = 8%.
The mark is better than the assessment of both diligence and talent = 18%.

Teachers of mathematics most often assess the performance of their pupils according to how they perceive the respective diligence and talent of their pupils. Similarly, the marks awarded most often correspond with their assessment of these factors. However, our research confirms that the marks ultimately awarded must also take into account other factors, most likely educational and school policy.

6 Conclusions

The conclusions to be drawn from our study suggest that there is a possibility of increasing the effectiveness of instruction via the enhancement of the professional competencies of the teachers by working with pedagogical-psychological data from the pupils. On the basis of a cursory comparison of research results in the Czech Republic and Germany, we can say that it is possible to use the data from the pupils not only for comparing the effectiveness of education systems, but also for comparing the professional competencies of the teachers in individual countries, especially their diagnostic and assessment competencies. The study also shows possibilities for the use of pupils' own feedback for systematic self-diagnostic analysis of teachers' performance, which is one of the preconditions of focused self-reflection. By such focused self-reflection, teachers can systematically increase their individual professional competencies in the mentioned areas.

Our results are also interesting from the point of view of assessment activities of the teachers and identification of their components (for example the comparison of marks and estimated performance with diligence and talent). In individual cases they could be a source of information for the teacher for comparing their own scheme of assessment with the schemes of other teachers – i.e., self-diagnostic data in this area.

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Symposium on Elementary Maths Teaching (SEMT)

Jarmila Novotná, Hana Moraová

Can we do more to support pre-service and in-service primary school teachers and their educators in their struggle for more effective, interesting and challenging mathematics teaching? Are 5–12 year old pupils' attitudes to mathematics and their needs and abilities different? Should more attention be paid to the teaching methods, contents and activities used in teaching these pupils? Does this group of pupils have any special needs? Are there any cognitive and psychological limits and boundaries that must be respected? What is their prior experience with mathematics which they enter primary school with and which can be built on? Are there any special tools and aids that can be successfully used for this specific target group? These are just some of the questions that made Czech primary school mathematics educators and researchers realize that the platform for the discussion of these issues was far from satisfactory as other international scientific events did not specialise in this age range, in consequence of which there was lack of space for discussing the questions specific for this age group.

The answer to this need is the conference *Symposium on Elementary Maths Teaching* (*SEMT*) which focuses on the teaching of mathematics to this group, i.e. children within the age-range 5–12 years.

SEMT is a biannual conference. As the 11th Symposium was held in August 2011, it is easy to calculate that the 1st SEMT took place in August 1991. A child, when conceived, has two parents. Our child, SEMT, also has two parents – two colleagues from Charles University in Prague, Michaela Kaslová and Jarmila Novotná. Conceived in 1990, SEMT was born in 1991 as the only conference focusing on the teaching and learning of elementary mathematics. SEMT has also had a number of aunts, uncles and friends supporting its development. The child has been gradually growing up, developing and assimilating new ideas, meeting new people, both from the Czech Republic and abroad, who helped its development.

Each SEMT focuses on one important central topic of elementary mathematics teaching. The development from general topics towards more specific problems of elementary mathematics teaching can be easily recognised from the main topics of all eleven SEMTs:

- 1991: The teaching of mathematics to elementary mathematics pupils
- 1993: The changing face of elementary mathematics
- 1995: Geometry and word problems for elementary mathematics
- 1997: Assessment and evaluation

- 1999: How the world of mathematics emerges from everyday experiences of children
- 2001: What is meant by the competence and confidence of people involved in the teaching of elementary mathematics
- 2003: Knowledge starts with pre-conceptions
- 2005: Understanding the environment of the classroom
- 2007: Approaches to teaching mathematics at the elementary level
- 2009: The development of mathematical understanding
- 2011: The mathematical knowledge needed for teaching in elementary schools

SEMT brings together elementary teachers, pre-service and in-service teachers and teacher educators and researchers from all over the world. This is reflected in both the participants attending and those giving lectures and workshops. The SEMT community has been gradually growing and the conference can now boast a relatively stable community of regular participants, while attracting the attention of newcomers from very distant corners of the world. The community involves colleagues from most countries in Europe, the Middle East, Japan, Australia and America. The mix of nationalities and the return of many old friends contribute considerably to the warmth and friendliness, which epitomises the SEMT conferences.

The multicultural background of the participants adds to the vibrancy of the symposium and the vigour of the interchanges between participants. This sharing of different perspectives on mathematics teaching has always been a strong and most attractive aspect of SEMT, as we gain greater understanding of our own practice through learning about the practice of others. The multi-nationality basis of SEMT conferences makes the participants realize that no problem or issue is a problem of just one institution or country. Discussions with fellow participants quickly make the participants realize that many of the most pressing matters are common for many countries. The sole chance to discuss the issues in this colourful community helps and sometimes even provides a solution.

The range of new ideas for helping teachers to make mathematics both a meaningful and an enjoyable subject presented and exchanged in the SEMT history is enormous. The number of formats for presentation, including plenary lectures, discussion groups, research reports, short oral presentations and posters provide enough space for both renowned and world-wide respected researchers and freshmen to share their ideas, research results and concerns. It has also helped new colleagues/post-graduate students to present their first papers to a discerning but appreciative international audience.

As on all international conferences, the contributions of the participants are published in the conference proceedings. Prior to this, there is a reviewing process in which all the contributions are reviewed by an international board. For those who attended SEMT, the proceedings act not only as a reminder of their SEMT experience. As the proceedings are available already at registration, the participants are given the chance to select among the different programmes in parallel sessions according to their interest. Moreover, the proceedings help those who do not feel very confident about their English to follow the oral presentations with textual support.

However, the proceedings are also meant for those who for any reason could not participate at the conference but are interested in the topic and the contributions. The number of copies of the proceedings has been gradually growing not only in reaction to the growing number of conference participants but also to answer the demand of the wider academic and research community. The evidence of the quality and reputation of the proceedings is its acceptation by Web of Knowledge in 2005.

SEMT has become an important international event with a high scientific as well as social standard. With the conference venue in the very centre of Prague, SEMT offers its participants unforgettable social and cultural experiences. The conference organizers are well aware of the fact that it is often during informal meetings that the ideas are born and so they pay equal attention to the organization of a conference dinner, a welcome party and a trip. Good food, live music, historical surroundings all foster nice atmosphere and prepare grounds for making friends and starting new cooperation.

It is hard to say how many times an event has to take place before it can be called traditional (it seems that it is generally agreed that to speak of a tradition of an event, it suffices if it takes place twice). However, there is no doubt that the biannual conference of the Symposium on Elementary Mathematics Teaching is now a well-established tradition as SEMT ’11 was already the eleventh conference.

The growing importance of contributions to the SEMT programme is also documented by the publication of the special issue of the *Mediterranean Journal for Research in Mathematics Education*, whose Volume 8, No. 1 in 2009 (guest editors Jarmila Novotná and Demetra Pitta-Pantazzi) contains augmented texts of selected plenary lectures from SEMT 2005 and 2007.

SEMT’s future looks promising. SEMT ’13 will be held in Prague at the Faculty of Education of Charles University in August 2013. Its theme is “Tasks and tools in elementary mathematics”. SEMT ’13 plenary speakers whose lectures we will have the pleasure to follow are: Olive Chapman (Canada): Engaging Children in Learner-Focused Mathematical Tasks; Rose Griffiths (United Kingdom): Working with children in public care who have difficulties in mathematics; Joanne Mulligan (Australia): Inspiring young children’s mathematical thinking through pattern and structure; Jennifer Young-Loveridge (New Zealand): What matters in mathematics learning to students: A tool for international comparisons.

We hope that the number of participants will yet again exceed one hundred and look forward to their valuable contributions, observations, remarks and ideas.

For a more detailed information about the conference see http://kmdm.pedf.cuni.cz.

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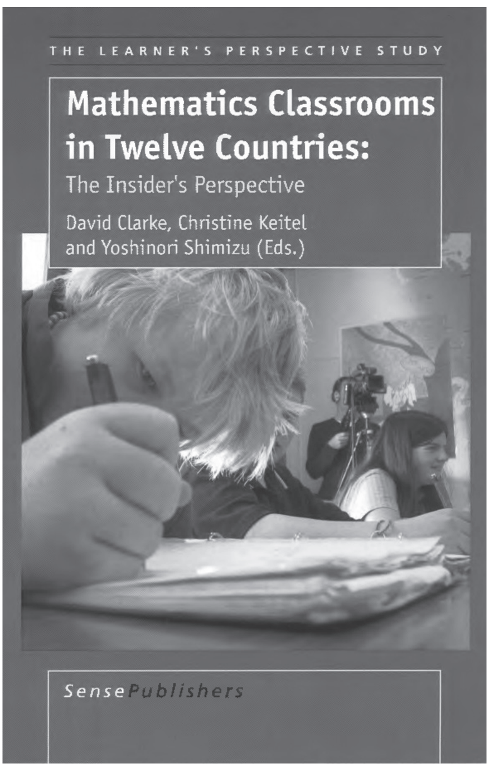
The Learner's Perspective Study: A Report

Jarmila Novotná, Alena Hošpesová

The Learner's Perspective Study (LPS) is the research direction aiming at international comparative research in mathematics education. The title stresses the intention to carry out teacher-focused studies by stressing the learner's perspective. An informal community of researchers from countries all over the world was formed around the project. The research framework was originally designed to realize research studies reporting on national norms of teaching practices with an in-depth analysis of mathematics classrooms in Australia, Germany, Japan and the USA. Eighth grade mathematics classrooms (students aged approximately 14) were chosen. Since its inception the project has grown, its purpose has been progressively reinterpreted and expanded. Research teams from other countries have continued to join LPS (for example, China, Sweden, New Zealand, Singapore, the Czech Republic).



A significant characteristic of LPS is its documentation of the teaching of a sequence of consecutive lessons. This feature enables to take into account the teacher's purposeful selection of instructional strategies. Another important feature of LPS is the exploration of learner practices. Technically, LPS methodology is based on the use of three video-cameras in the classroom supplemented by post-lesson video-stimulated interviews. What is vital here is that all interviews be held immediately after the lesson. They enable the revelation of the teacher's beliefs and comparison with students' perception and researchers' view.

The work of LPS community is reported in a series of books published by Sense Publishers. In this text, we will report on two of them.

International comparative and cross-cultural research emerging within the community offers the insights into the practices employed in different school systems, can help to identify common values and shared assumptions, causes to question and revise the assumptions about own practice and the theories on which this practice is based. All these features could be found in the book *Mathematics Classrooms in Twelve Countries. The Insider's Perspective* edited by David Clarke, Christine Keitel and Yoshinori Shimizu. The book brings various portraits of classroom practice from twelve countries participating in LPS. The insider perspective is given by the fact that the authors of the chapters are insiders in the country, in the cultures and school systems, and did their analysis from their positions.

The book provides different views on school practices in mathematics.

- Observation of teaching practices from the point of view of the teacher – pupil cooperation and its cultural specific.

Keitel (in a chapter *'Setting a Task' in German Schools: Different Frames for Different Ambitions*) looks closer at mathematics classroom practice in Germany to find out what kind of tasks are set, what are the differences in the ways teachers set tasks and if, and how, these differences may affect students' learning. Similarly Hino (*The Role of Seatwork in Three Japanese Classrooms*) examines the reality of the teacher's support for individual students during seatwork (Kikan-Shido in Japanese).

- Identification of connections between the learner's and the teacher's actions.

Begehr examines how students participate in mathematics lessons and current status of students' verbal communication (*Student's verbal Actions in German Mathematics Classes*). Williams studies student construction of valued social and mathematical meaning supported by particular documented teacher and learner practices (*Autonomous Looking-In to Support Creative Mathematical Thinking: Capitalising on Activity in Australian LPS Classrooms*). Huang, Mok and Leung in their common chapter characterize, examine and compare the practice problems in Hong Kong, Macau and Shanghai schools (*Repetition or Variation: Practising in the Mathematics Classroom in China*).

- The use of variety of teaching approaches by individual teacher in the course of teaching a lesson sequence.

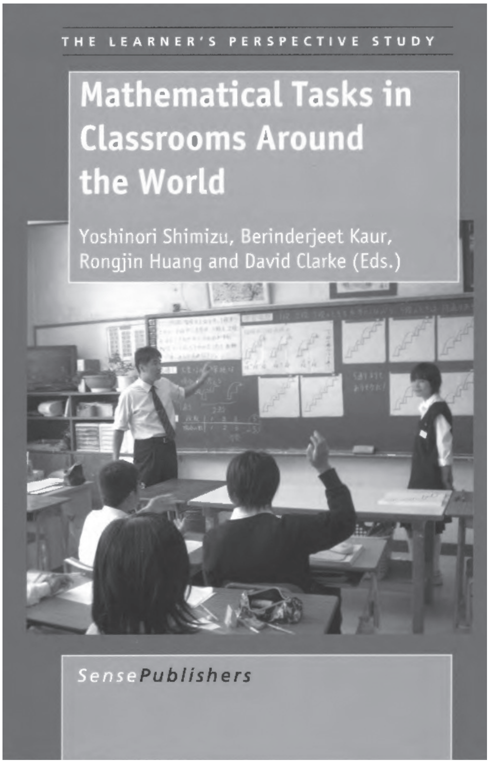
Kaur, Low and Seah explore the roles of textbook and homework from the teacher and student perspective (*Mathematics Teaching in Two Singapore Classrooms: The deemed Role of the Textbook and Homework*). Park and Leung characterise effective practice in the Korean mathematics classroom (*Mathematics Lessons in Korea: Teaching with Systematic Variation*). Emanuelsson and Sahlström showe how students and teachers in two Swedish classrooms constitute learning in interaction (*Same from the Outside, Different on the Inside: Swedish Mathematics Classrooms from Students' Point of View*).

- Characteristic of competent teaching practice from different national and cultural perspectives.

Mok presents the teacher's and the students' perception of their mathematics lessons based on the analysis of the post-lesson interviews. She integrates both views with what happens in the classroom and discusses the nature of the teacher-dominating lesson (*Teacher-Dominating Lessons in Shanghai: The Insiders' Story*). Binterová, Hošpesová and Novotná identify a sequence of several constructs from the theory of didactical situation in the Czech lesson (*Constitution of the Classroom Environment*). Mok and Real discuss some interesting patterns of similarities and differences between teachers of the same region (*A Tale of two Cities: A Comparison of Six Teachers in Hong Kong and Shanghai*).

– The relationship of the teacher's and the learner's practices.

Shimizu discusses the results of the analysis of post-lesson video-stimulated interviews with the teacher and the students (*Discrepancies in Perceptions of Mathematics Lessons between the Teacher and the Students in a Japanese Classroom*). Fried and Amit describe two dichotomies, private versus public and collaboration versus authority, and show their relation (*The Israeli Classroom: A Meeting Place for Dichotomies*). Gallos (*Students' Private Discourse in a Philippine Classroom: An Alternative to the Teacher's Classroom Discourse?*) reacts to the typical situation in Philippine classrooms of a teacher doing most of talking and focuses on the alternative of students talking privately in mathematics classrooms.



– The implications for teacher education and the organisation of schools.

Ulep examines the consequences of the teacher's motivational strategy (*'Ganas' – A Motivational Strategy: Its Influence on Learners*). Wood, Shin and Doan (*Mathematic Education Reform in Three US Classrooms*) analyse the data collected in US classrooms for the purpose of examining how these classrooms realise the goals of the reform in school mathematics instruction.

In the end of the book, the reader can find short descriptions of the school systems in the participating countries, and Authors Index and Subject Index.

The second book with the title *Mathematical Tasks in Classrooms around the World* was edited by Yoshinori Shimizu, Berinderjeet Kaur, Rongjin Huang and David Clarke. As it is clearly indicated in the title, it focuses on the fundamental part of school mathematics, mathematical tasks. Mathematical tasks are crucial mediators between mathematical content and the mathematics learner. "Doing" mathematics includes solving mathematical tasks as an important component of all activities. For every task, there exist pieces of knowledge that enable to solve it. Not all of them learners have at their disposal. Their learning can be seen as an extension of the repertoire of means for solving assigned tasks. The teacher's task is to create a suitable environment for such an extension.

The classroom implementation of a task is a synthesis of task, teacher, students and situation. As mentioned in the abstract of the book, "of particular interest are differences in the function of mathematically similar tasks when employed by different teachers, in different classrooms, for different instructional purposes, with different students".

The book consists of ten chapters, the Appendix describing the LPS research design, and Authors Index and Subject Index. Each chapter deals with mathematical tasks from a different perspective. The first chapter written by the book editors *The Role of Mathematical Tasks in Different Cultures* brings general theoretical perspectives concerning the role of mathematical tasks in the educational reality. It covers the centrality of tasks in mathematics classroom instructions, the relationship between mathematical tasks and LPS, the nature of mathematical tasks in classrooms, the theoretical alternatives in considering their classroom use, and cognitive demands of different tasks for different learners.

The other chapters deal with various aspects of mathematical tasks against the background of one or more countries. Five of them are based on the situation in one country, three are devoted to the comparison of the situation in two countries and the last, tenth chapter compares aspects of the theme in five different countries.

Let us first look at the chapters devoted to the situation in one country. Each chapter focuses on a different perspective.

- In the second chapter, *A Study of Mathematical Tasks from Three Classrooms in Singapore*, Kaur focuses on the source and nature of mathematical tasks used by three competent teachers in the grade eight classrooms in Singapore; she analyses and compares data from various perspectives.
- In the third chapter, *Mathematical Tasks as Catalysts for Student Talk: Analysing Discourse in a Norwegian Mathematics Classroom*, Bergem and Klette deal with their data from the perspective of communication in the ninth grade Norwegian classrooms. The sort of mathematical reasoning when talking mathematics through tasks from the everyday settings, the negotiation and stabilization of distributed expertise among learners, the nature and the scope of mathematics tasks involved and the teacher's support influence are studied.
- In the fourth chapter, *Engaging Students with Mathematical Tasks in a Large Class*, Cronberg focuses on what students from the Year 8 in the Philippine public secondary schools do during the group work in large classes and what these tasks require from students with the main attention paid to geometric proofs.

- The fifth chapter, *A Task-Specific Analysis of Explicit Linking in the Lesson Sequences in Three Japanese Mathematics Classrooms*, written by Shimizu, is based on the important theoretical construct "linking" and analyses its influence on the mathematics teaching and learning in Japanese schools dealing with three topics: Linear equation, Plane geometry and Number and expressions.
- In the sixth chapter, *Linking in Teaching Linear Equations – Forms and Purposes: The Case of the Czech Republic*, Novotná and Hošpesová extend the study of linking by introducing a finer classification of its types. The influence on the teacher's and students' behaviour is analysed; illustrations come from the eighth grade classroom episodes from two Czech schools.

Three chapters are devoted to comparative studies of various aspects of mathematical tasks in different countries. Via the comparison of the lessons between two different countries, the important properties of mathematical tasks and their implementation in mathematics lessons are highlighted.

- In the seventh chapter, *Comparison of Learning Task Lesson Events between Australian and Shanghai Lessons*, Mok concentrates on the possible contribution of learning task lesson events to the building of relationships between procedural and conceptual knowledge. It can also be seen as an example how an event-coding technique can be successfully combined with direct observations of a learning unit.
- The mathematical tasks that are dealt with by Huang and Cai in the eighth chapter *Implementing Mathematical Tasks in US and Chinese Classrooms*, are questions, problems, applications and exercises. Data from one school in China and one school in the USA are analysed and compared. Factors associated with the implementation of mathematical tasks in these schools are studied from the perspective of maintaining and reducing cognitive demands and serve as the basis for characterising cognitive demands of mathematical tasks.
- In the ninth chapter, *Student-Created Tasks Inform Conceptual Task Design,* Williams studies the types of questions students ask themselves to achieve a deep understanding. The analysis is based on the comparison of studies in Australia and in the USA. Pedagogical advantages of integrating student-formulated questions that elicit complex thinking are documented.

The last chapter, *A Functional Analysis of Mathematical Tasks in China, Japan, Sweden, Australia and the USA: Voice and Agency*, written by Mesiti and Clarke, represents an example of a comparative study including more than two countries. It focuses particularly on differences in the function of mathematically similar tasks when employed by different teachers, in different classrooms, for different instructional purposes and with different students; the significance of differences between social, cultural and curricular settings, classroom communities are taken into account.

The series of LPS books published in Sense has not finished. Further topics are in diverse stages of preparation for publication: international perspectives on the teaching and learning of algebra; competent teachers in mathematics classrooms around the world; coherence in the mathematics classroom; difference in mathematics classrooms around the world or students' voice in mathematics classrooms around the world.

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