ACTA UNIVERSITATIS CAROLINAE

PHILOSOPHICA ET HISTORICA 2/2017

MISCELLANEA LOGICA

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CHARLES UNIVERSITY • KAROLINUM PRESS • 2017

Editor: Michal Peliš (Charles University)

This volume was supported by Progres Q14 Rationality Crisis and Modern Thought.

http://www.karolinum.cz/journals/philosophica-et-historica

ISSN 0567-8293 (Print)
ISSN 2464-7055 (Online)

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LOGICAL SPACE AND THE ORIGINS OF PLURALISM IN LOGIC

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ABSTRACT
The fact that there is a plurality of systems that we call logics makes it requisite to attempt an explanation and thorough evaluation of the role of logic. I exploit the analogical development towards the pluralism of geometry to show that both disciplines are about some kinds of space which they explicate and that we can choose with some freedom the tools for engaging in an enterprise of these disciplines. After revisiting the development of non-classical (i.e. non-Euclidian) geometries, I present logical expressivism, as coined by Robert Brandom, and, returning again to geometry, show that an analogous doctrine of geometrical expressivism can also provide a viable account of the nature and purpose of the discipline and the reasons for plurality of both geometries and logics.
Keywords: pluralism, expressivism, logical space, geometry, holism

Today we have already gotten well used to there being a plurality of logics and so it seems difficult to understand the approach of logicians before the twentieth century who were convinced that there cannot be more than one logic. Pluralism is a sign of the fruitful development that logic as a mathematical discipline has undergone. Every new logic has the potential to show us hitherto unknown possibilities of the mathematical methods which form the backbone of various logics. Besides being interesting in themselves, new logics help us see particular properties of the already established ones. The development that led to the plurality of logics surely had its rationality and has brought many interesting results, as well as enriched logic and many germane disciplines.

There are, therefore, good reasons to be happy about this pluralism and not to regard it as something which we should be bothered by. Yet, I think it still is a remarkable phenomenon which has to be philosophically reflected upon. Although we have gotten used to plurality, we have to admit that there is something paradoxical about it which cannot be so easily dismissed on the grounds that I just mentioned. We are used to all kinds of pluralism in various disciplines, yet plurality should have some common denominator and therewith also some limits[1]. And what should be clearly delimited if not logic? People may have various opinions on issues that they get to discuss, yet the very rules guiding the discussion should be rather firm if that discussion is supposed to be possible at all. And it is quite natural to think that logic should be about the rules of correct argumentation, of inferring conclusions from some premises that we agree on (such an opinion is

[1] Note that even Beall and Restall who largely helped make the idea of *logical pluralism* popular also call for imposing limits on it (see Beall (2006)).

https://doi.org/10.14712/24647055.2017.12

not only expressed in textbooks on critical thinking but also in textbooks on logic itself). The plurality of logics thus seems to indicate uncertainty about these rules, at least in the case that pluralism has no clear limits and we cannot say what can be regarded as logic at all. The naive suggestion would be that we are facing a dilemma. Either logic has somehow gone astray and is not investigating these rules of correct argumentation anymore or, worse, the plurality of logics reflects that no such rules exist (anymore?) and therefore rational debate is no longer possible[2].

The second possibility is preposterous, as rational discussion obviously is possible, even if it can get very complicated. Yet, the idea that logic might not be pursuing its real purpose is worthy of investigation. If the discipline describes something which turned out to be quite apt to change – as for practically any logical law we can think of a logic according to which it holds as well as of one with an opposite verdict – then it is probably failing to describe the very foundations of our cognition, as we can expect such foundations to be something quite stable and immune to change, whether it be foundations in civil engineering or in epistemology. So if logic can be changing its statements almost at almost the same rate as the most empirical sciences, it is probably instead only describing some quite parochial part of our epistemic apparatus. Yet, perhaps we should rethink what we regard as the purpose of logic. Or maybe we could try to somehow distinguish, among all the so called "logics", the real ones (or, even better, the real one) which truly describe the foundations of our rationality, from those mathematical systems which do not deserve such a name. Nevertheless, let us investigate for a moment two disciplines that underwent a development that was, to a considerable degree, similar to that of logic. Namely, geometry and arithmetic.

1. Pluralism in geometry and arithmetics: comparable to the case of logic?

For the moment, we will not try to specify with particular precision when two logics are rivals and how much they have to differ from each other in order to vindicate logical pluralism, should both of them be accepted as legitimate logics. There are many ways in which logics can differ. Susan Haack (see Haack (1978)) distinguished between cases when logics are indeed *rivals* (or *deviant*) with respect to one another, and the case when one of them is merely an extension of the other one. For example, intuitionistic logic is a rival of classical logic, as it shares the same language and yet pronounces different verdicts as to the validity of logical laws. On the other hand, normal modal logics (or second order logic) are mere extensions of classical logic, i.e. in the restricted language the same laws hold. I want to understand two logics as being different irrespective of Haacks's distinction between rivality and extension. The dispute about whether the law of the excluded middle holds universally is for me on the same footing as that of whether the modalities belong within the purview of logic. Intuitionistic logic, classical logic and the modal logic S5 are simply different logics in my understanding. I will purposely leave the problem of individuation of a logic and therewith that of logical pluralism in this

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2 Some think that anthropological research of inference rules in exotic cultures proves as much, see Triplett (1988) for a short discussion thereof.

rather naive and undeveloped shape, as I want it to be specified and elaborated later in the course of our investigation of the role of logic.

However we interpret it, the plurality of logics is a relatively recent phenomenon. That there is only one logic was taken for granted to the point of it not being worth mention[3]. Kant attempted to show that there are good reasons for logic to be the way it was. In fact, his systematization of logical judgments presented in the *Urteilstafel* serves as one of the foundations of his transcendental philosophy. Thus, logic in its specific form is necessary for the whole complex building of our rationality. Of course, when you start giving reasons for something that is taken for granted, you, even if only in a long run, encourage the attempts to cast doubts on these reasons and consequently also the platitudes that they were supposed to substantiate. Thus, we will soon see that Kant's arguments for why logic was the way it was also opened the possibility of countering them and eventually awake the suspicion whether logic cannot, in fact, be changed. But first let us digress a little bit from logic to geometry and arithmetic, as both Kant's position and its subsequent development display many illustrative analogies to the case of logic.

According to Kant human cognition has two sources, sensibility and understanding[4], and logic describes the fundamental features of understanding, i.e. those features without which there would be no understanding at all. Analogously, in many ways , he sees geometry and arithmetic as fundamental for our sensibility (the study of which he calls *aesthetics*), which is our ability to receive the raw materials that our understanding can work with. Let us consider to which degree we can speak of pluralism in these other allegedly fundamental disciplines and where it can stem from. We hope to achieve a better understanding of logical pluralism by considering the possible analogies between the development logic, arithmetic and geometry underwent. It is very useful to make Kant our point of departure, as he can be seen as an advocate of monism in all of these disciplines. Pluralism therefore had (and despite all the progress still has) to develop in opposition to his views.

1.1 Pluralism in arithmetics?

As is well known our inuition has, according to Kant, two basic forms which enable us to perceive any object at all – namely, space and time. And time, characterized by its one-dimensionality, is described by arithmetic, the science of number. Arithmetic shows us the necessary features of our perception of time. Despite all the controversy which this Kantian doctrine of arithmetic caused, we have to say that it does not help us to any illuminative analogy with the problem of logical pluralism because arithmetic has developed differently from logic since Kant's time and the controversy was not about the possibility

[3] This does not mean that there were historically no disputes about logic. For example, the Stoics developed a logic which differed from Aristotle's. But while on the one hand, Aristotelian logic was historically dominant, on the other hand logical pluralism as the thesis that there are more legitimate logics was still hardly seriously considered, as even the Stoics though that there is only one logic which Aristotle, unlike them, failed to describe successfully.

[4] Nur so viel scheint zur Einleitung, oder Vorerinnerung, nötig zu sein, daß es zwei Stämme der menschlichen Erkenntnis gebe, die vielleicht aus einer gemeinschaftlichen, aber uns unbekannten Wurzel entspringen, nämlich Sinnlichkeit und Verstand, durch deren ersteren uns Gegenstände gegeben, durch den zweiten aber gedacht werden (Kant (1954), A16/B30).

of developing a different arithmetic. Kant's view provoked attempts to show that this discipline has a significantly different status (most prominently Frege's logicist project) than he thought, but it did not lead to pluralism. Nobody considered the possibility of there being more possible arithmetics[5].

Clearly the connection of arithmetic with time is controversial and the idea of the linear shape of time is something which was attacked by both physicists and philosophers[6]. Furthermore, the classification of arithmetic knowledge as synthetic a priori was attacked by Frege, who launched his logicist programme in an attempt to refute it. To what degree such a programme can succeed is still disputed and these disputes are actually closely related to the problem of logical pluralism, because if we try to verify or refute the logicist thesis that arithmetic can be derived from logic, it is, of course, important to know what logic is (or, put otherwise, which logic it is that are we trying to show contains arithmetics). Some authors (notably Ian Hacking (see Hacking (1979)) or Gila Sher (see Sher (1991)) have tried to legitimize their preferred logics as leading to their preferred verdicts regarding the viability of logicism.

And yet, as we said, there is no direct analogy to logical pluralism in the form of arithmetical pluralism. There are more axiomatisations of arithmetic, such as the Peano, Robinson and Presburger axioms, yet these cannot be said to describe different arithmetics[7]. These sets of axioms are distinguished mainly because they serve as different tools for investigation of the logical properties of arithmetic, differing in two basic features. On the one hand, they can have neat formal properties which make them easier to treat (as the Presburger or Robinson axiomatic), on the other hand, they can come closer to describing the actual practice of arithmetic as Peano arithmetic or, even if we allow for a shift in logic, the second order Peano arithmetic which describes the structure of natural numbers which underlies our everyday aptitude to count and in general deal with numbers l categorically.

One more aspect of the modern treatment of arithmetic may resemble pluralism, namely the fact that, for example, the first-order Peano arithmetic has got many non-isomorphic models, i.e. it is not categorical. Do these models perhaps show us that there are different kinds of arithmetic, different ways of understanding numbers and their properties and interactions? Well, hardly. The non-standard models of Peano arithmetic (and the even more non-standard ones of Robinson arithmetic) are surely an interesting and fascinating object of mathematical study, yet they can hardly be said to constitute alternative arithmetic which could, under some circumstances, supersede the arithmetic based on the daily practice of counting and related activities. Again, just as with the plurality of arithmetic theories, the plurality of models of a given theory primarily illuminate the epistemological status of arithmetic, in particular the acceptability of logicism or formalism. The difference between the plurality of theories and that of models, speaking

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5 This is not to say that there were no controversies during the history of the discipline, e.g. about the properties of zero, whether it is a number at all, etc. Yet no arithmetic pluralism has ever really arisen.

6 Among them we can mention Einstein who showed that a different idea of time is possible and even preferable to the simplistic one. Besides this, temporal logicists model time as branching.

7 Unless we see the matters from a radically formalistic point of view which would claim that every set of axioms constitutes a specific object of study (with equivalent sets of axioms constituting the same object, of course).

somewhat figuratively, is that while we tend to actively create the theories, the models are rather found by us in course of examining the properties of these theories.

Though Kant's views of arithmetic are controversial and may be legitimately questioned, we cannot say that the controversies that ensued led to arithmetic pluralism, analogous to logical pluralism. The story, though, gets more intricate with respect to geometry, as this discipline in fact developed – to a great degree due to the discussions initiated by Kant – towards a pluralism resembling the one in logic.

1.2 Pluralism in geometry

The development in geometry that led to the problematization of Kant's view of it and opened the possibility of pluralism preceded a similar process that logic was to undergo. I offer an overview of the revolution which the discipline underwent since the advent of non-euclidian geometries in Arazim (2013). For our purposes a short summary will suffice. The gist of the story is, as many have probably already guessed, that Kant thought that there is only one possible geometry while the subsequent development of the discipline proved otherwise.

Just as is the case with time, space is, according to Kant, a fundamental form of our sensibility, of our intuition, i.e. the ability to be given objects of cognition which then serve as the material to which we apply our cognitive capacities. Now, just as arithmetic describes the structure of time, so does geometry describe the structure of space. It is fundamental that space, the same as time, is not seen by Kant as an entity independent of us as, e.g. a table can be, but rather as something we create at least in the sense that it could not exist without us, something which we have to constitute to make perception possible. Yet, despite this active element in regards to space, it still seems that we do not have any freedom as to how we construe it. There is just one shape it can take and this is revealed to us thanks to our pure intuition. Euclidian geometry provides a rigorous exposition of this structure. This discipline is possible because we are endowed with the ability to have intuitions and hence to operate within the forms of sense (i.e. time and space).

Yet there was an old controversy about the fifth postulate of Euclidian geometry. One of several equivalent formulations of it is that, given a line and a point outside of it on a plane, there can be only one line on the same plane going through that point and never crossing the original one (which we then call its parallel). In the course of the nineteenth century, it was shown that you can create both a theory in which an infinite number of such lines can be drawn as well as a theory in which no such line at all can be drawn. Both subsequently earned the title of geometry[8], the first one became known as *hyperbolic geometry* and the second one as *elliptic geometry*. It turned out that these theories were not only consistent, but that they could also be reasonably applied to space (or more exactly – to our discourse about space and experience thereof, as we are acquainted with it independently of the adoption of one of the specific theories of geometry).

Elliptic geometry can be seen as a theory which describes the behaviour of lines on a surface of a sphere (a plane with 'positive' curvature), while hyperbolic geometry deals

8 For an explanation of how this became possible, I refer the reader again to my article Arazim (2013).

with lines on the surface of a kind of valley (plane with 'negative' curvature). It is natural to feel that Euclidian geometry remains somewhat more fundamental than the others and that these other geometries are based on stretching the notion of line in an illegitimate manner. Yet, from the point of view of those geometries, it is Euclidian geometry that does not deal with real lines, but with the geodesics on a curved surface[9].

The important point for us is that in geometry we can definitely speak of pluralism. This pluralism is not as broad as the logical version (we have presented only three axiomatisations which can be legitimately used for explication of space and therefore called geometries) because there are not as many geometries as logics, yet it is also present in geometry. And it is just as troubling, particularly for someone who accepted the Kantian view regarding the character of geometrical knowledge. How can the discipline be apodictically certain if there are more possible answers to some of its fundamental questions? Obviously the emergence of pluralism forced theoreticians to seek a deeper philosophical explication of the nature and purpose of the discipline. And that is exactly what I claim to be necessary for logic, as well. At this point it will be useful to review a few of the approaches to the plurality in geometry which suggest themselves and which in fact were adopted by theoreticians of various backgrounds.

1.3 How to react?

The emergence of alternative geometries was certainly shocking. No doubt for someone who is getting to understand the development of the new geometries it at first glance also appears very surprising, as it strikes one as quite strange that such a fundamental concept as, for instance, the line can be understood in radically different ways. One of the natural reactions to a shocking event is denial, to try and convince oneself and others that it did not really happen, to try and explain it away. Indeed, many Neo-kantians in the nineteenth century tried to do exactly this, as is well recounted in chapter 3 of Coffa (1993).

In this case one, can understand what motivated many theoreticians to adopt this conservative stance. We have already mentioned that alternative geometries seem to be based mainly on stretching the notion of line. In fact, the situation reminds one of Quine's critique of those who attempt to change logic; namely, that they merely succeed in changing the subject (see Quine (1986), p. 81). After all, Beltrami, who has shown how the lines of non-classical geometries can be understood as Euclidian geodesic lines on a curved plane, intended himself to show that the only legitimate geometry is Euclidian, as the others do not really speak of lines in the proper sense of the word.

This traditionalist approach may still have its appeal. We also have to admit that, at least psychologically, Euclidian geometry probably has to be the first geometry one learns to work with. The understanding of what a line is and subsequently all the other notions, such as triangle, circle, etc., is acquired in a much simpler manner if the first geometry one learns is Euclidian. Quite similarly, note that some logics are clearly more basic from a similar psychological perspective. Many logics have an intuitive appeal, perhaps that of

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9 Compare this with the difference of perspective of someone who understands the operation of addition in a standard manner and somebody who understands as a non-standard arithmetician of the kind described in Kripke (1982).

the syllogistic is particularly high. Yet, putting the syllogistic aside and considering the modern logics that came after it, it has to be said that, analogously to Euclidian geometry, classical logic is, by far, learned most easily by somebody who is getting acquainted with modern logic in general. It is no accident that the other logics – intuitionistic, the modal ones, etc. – are typically learned subsequently, in more advanced courses, as variations on and expansions of the theme of classical logic, which in many ways behaves in a much more orderly fashion and is easier to work with overall[10].

Yet, this psychological privilege of Euclidian geometry and classical logic, as important as it is for teaching either geometry or logic, could perhaps be circumvented by trying new pedagogical approaches to the alternative systems. Be that as it may, psychological facts are not truly important and fundamental for the philosophy of either logic or geometry. The question is instead whether the Euclidian geometry is prior to the other ones in some transcendental sense, i.e. whether the very notions of the non-classical geometries can be understood only through the perspective of the Euclidian one. And no argument in favour of such a thesis is at hand. Indeed, when thanks to the works of Eugenio Beltrami and Hermann von Helmholtz, the alternative systems were shown to be geometries in the sense of being capable of describing space, there seems to be no rules for somehow a priori privileging one of them, for seeing it as being somewhat more basic. Indeed, Helmholtz argued exactly that what seems to be the correct lines, the Euclidian ones, is seen as curved by those who live in, say, a world with a space not obeying Euclidian rules[11]. Here, we come to another approach to the plurality of geometries.

1.3.1 Empirisation of geometry

The first approach to the plurality of geometries consisted in strongly preferring the already established system to the emerging ones. The approach which we are going to consider now differs from the conservative one in that it does not privilege Euclidian geometry. On the other hand, it wants to get rid of the plurality just as the conservatives wanted. Hermann von Helmholtz, a German theoretician active in the second half of the nineteenth century, tends to switch between empiricist and holistic views, or at least varies the emphasis in his writings on geometry. As we already mentioned, we can find passages where he focuses on the relativity of the fundamental notions. By thought experiments he makes the reader see how a world can be imagined in which what seems to be perfect lines obeys either the elliptic, or the hyperbolic laws (or the Euclidian ones, of course). Elsewhere, however, Helmhotz seems to suggest that geometry should be regarded from now on as empiricial science[12].

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10 It is, for example typically much simpler to find out whether a given formula is a tautology by the means of the truth-tables than by the means afforded for intuitionistic logic.

11 For an exposition of the important arguments and thought experiments due to Belmtrami and Helmholtz, see again the third chapter of Coffa (1993) or their original writings, namely Beltrami (1868) and Helmholtz (1870).

12 Nehmen wir aber zu den geometrischen Axiomen noch Sätze hinzu, die sich auf die mechanischen Eigenschaften der Naturkörper beziehen … dann erhält ein solches System von Sätzen einen wirklichen Inhalt, der durch Erfahrung bestätigt oder widerlegt werden, eben deshalb aber auch durch Erfahrung gewonnen werden kann (Helmholtz (1870), p. 25).

The empiricist proposal is remarkable. When Kant asserted that geometry is synthetic a priori, the development of non-contradictory alternative systems seemed to force us rather to abandon the *synthetic* part of Kant's classification of geometry as *synthetic a priori*. We will get back to this attempt later on. Helmholtz was nevertheless pioneering the refutation of categorizing geometry as *a priori*. He insisted that the acceptance of geometrical theories should be based on empirical findings. The truth of the fifth postulate can be, according to this approach, tested, e.g. by testing the equivalent thesis that the sum of angles in a triangle equals two right angles. Thus, we can focus on measuring the angles of triangles as precisely as we can in order to establish which of the geometries indeed describes reality the way it is.

This position opens new approaches of seeing how we form our theories, yet it is also very problematic. Let us begin by pointing out its shortcomings. Testing a thesis by empirical measurement seems like a relatively straightforward procedure. Yet, as was shown in Kuhn (1962) and Quine (1951), testing even relatively parochial hypotheses, i.e. close to the margins[13] of the Quinean web, can be a relatively complicated process. What appears to be recalcitrant empirical data can be explained away in many cases. One can be stubborn enough to declare such experience a mere hallucination. The process of empirical testing is, of course, much more complicated for theses which belong more to the center of the web. Quine himself, to be sure, adds that as anything can be saved, anything can be sacrificed as well[14]. Yet at a certain point, when we endeavour to empirically test some of the really core beliefs and principles, the empirical testing becomes unintelligible rather than just very complicated because some of the core beliefs can hardly be treated as something which it makes sense to question, as they enable the enterprise of testing or even of pursuing truth in the first place.

In the case of geometry it is very implausible that we should ever consider, say, any empirical finding concerning the sum of the angles in a triangle more trustworthy than one or another geometrical theory we adhere to. To be able to perform this measurement, or a similar one, we surely have to be equipped with some tools, such as a ruler. This ruler has to convince us that what we are given is indeed a triangle. Yet, to ascertain that, for instance, the sides of this triangle are indeed not curved, we first have to verify that the ruler is itself not a little bit curved. Thus, we get into an infinite regress[15]. The idea of deciding between the geometries by empirical measurements is, therefore, ill-founded. No further development of the tools at our disposal can bring about a change of this simple fact.

Does this all mean that the idea of empirical findings playing a role in our acceptance of geometry is just a mistake committed by Helmhotz and other authors? As was already mentioned, in *Two dogmas* Quine asserts that anything can be refuted in the light of empirical findings. Helmholtz has, I believe, pointed in the right direction, but we have

[13] By *margins*, I mean, as can be expected, those parts of the web we are most prone to adjust and thus consider as very empirical.

[14] He even explicitly mentions logic as something we can review in the light of empirical data. He would hardly think otherwise about geometry in this respect.

[15] This is an idea coming from Henri Poincaré. A nice exposition of how he developed his opinions can be found in Shapiro (1996).

to examine more closely what it means to refute something on the basis of empirical findings.

1.4 Holistic approach to geometry

In order to empirically test a hypothesis one needs a lot of preliminaries. First of all, it is necessary to understand properly what the hypothesis asserts, i.e. to know its meaning. An important part thereof is to understand what would count as a refutation and what as a confirmation of the hypothesis. This requires that we have a solid theoretical background on which we can base our experiments. Only with such a background does it make sense to say that a given proposition was shown to be true or otherwise. These are not particularly surprising facts, yet oftentimes people fail to realize them.

We have just sketched an argument in the section above for the thesis that we in fact cannot, based on empirical data, really construct the necessary framework to be able to assert that one of the geometries is right or wrong. Put otherwise, we cannot reasonably ask, whether *real space* is Euclidian or otherwise. We thus cannot move from the Kantian *synthetic a priori* to *synthetic a posteriori*. Would it perhaps be possible to move instead towards *analytic a priori*? That could mean lots of different things, as the Kantian notion of analyticity, as is well documented in Coffa (1993), allows for multiple interpretations. If we say that geometry should depend on the meanings of words, we have to be careful. The meanings of the fundamental geometrical terms are not clear enough to help us judge which of the geometries is right. Should one perhaps feel that the Euclidian geometry expresses the notion of a triangle most correctly, then it would not be clear why this feeling should be taken as more than just a matter of personal preferences and idiosyncracies. In fact, the development of non-Euclidian geometries showed exactly that what is objectively determined in the geometrical concepts in our standard everyday use of them is just that which leaves the dispute between geometries undecided at the analytical level.

As was shown, e.g. by Helmholtz, if one of the geometries is legitimate, then so are the others because all of them describe space though each is based on a different understanding what space is (that is, a space with either positive, negative or neutral curvature). It is not the case that one can say that one of the geometries is true while the others are false. In fact, each can interact with physics, yet each demands physics to adapt to it - Euclidian geometry, for instance, has to be paired with a different mechanics than the elliptic one (for a more detailed description, see again Arazim (2013)). Thus, any of the geometries can be used as a valuable tool for creating broader theories that help us to understand the world better. More should be said about how these geometries specifically do this. When we achieve such an explanation, we will be in a position to understand what geometry is and what purpose it serves (contrasted with that of, e.g. mechanics) and to have an enlightening account which not only is not in conflict with the plurality, but shows why it arises. Let us try to arrive at that with the help of getting back to investigating pluralism in logic.

2. Accounting for the plurality of logics

Having reviewed pluralism in geometry we will now try to use what we have learned to help us towards an account of the plurality of logics. Eventually we hope to move towards an overall picture which will advance our understanding of both logic and geometry more than what we achieved in the previous section. Thus, we travel on the path of analogy from geometry to logic and back, always achieving some progress in understanding the respective discipline.

We began by noting that the plurality of logics can be seen as something quite disturbing (despite the mathematical benefits which, as I emphasized at the very beginning, it brings). Somebody not familiar with the modern plurality would probably expect logic, even more than geometry, to be a body of truths we cannot doubt without losing any certainty we could have and ultimately even blurring the epistemic notions such as that of doubt, certainty, knowledge, etc. We reminded oureselves of Kant as a philosopher who believed that logic (or what was considered to be logic in his time) does not by any means have its form by accident, indeed that no change thereof is possible.

Here we are clearly not just speaking about a perspective someone could have had before the rise of the many modern logics and thus not about a view which is only of note in regards to the history of the discipline. Attempts at monism were undertaken much more recently. The debate between intuitionistic and classical logic was led to great degree by monists – especially the intuitionists were convinced that they are presenting the correct logic. Later, however, the two logics got along better and have coexisted relatively peacefully for quite some time now[16], though Michael Dummett still insisted that intuitionism was not compatible with classical logic, particularly not with acceptance based on the thesis that these logics speak about something else, a position adopted by Quine and which we will discuss presently. Intuitionism, as Dummett states, consists precisely in seeing classical reasoning as illegitimate[17].

An advocacy for monism was provided by Quine (see Quine (1986)), who used a modification of his own *gavagai* argument for the purpose of refuting the possibility of logical pluralism. While that original argument (used by him on various occasions, among others in Quine (1960)) purported to show how much has to remain undetermined in translation, the variant thereof regarding logic essentially does the contrary. According to Quine, logic is a body of truths so basic that we have to impose them on somebody we are translating. As a consequence of the celebrated *principle of charity*, we have to reject a translation which renders someone as disagreeing with us about the laws of logic because it violates the maxim of translation to *save the obvious*. In spite of all the leeway that we have while translating, the logical laws and truths are unshakable for Quine, as he urges

[16] About the coexistence see, e.g. Dubucs (2008), p. 50: "... times where controversy was raging are disappearing from collective memory."

[17] "As Kreisel has emphasized, the intuitionistic philosophy of mathematics comprises two theses: a positive one and a negative one. The positive one is to the effect that the intuitionistic way of construing mathematical notions and logical operations is a coherent and legitimate one, that intuitionistic mathematics forms an intelligible body of theory. The negative thesis is to the effect that the classical way of construing mathematical notions and logical operations is incoherent and illegitimate, that classical mathematics, while containing, in distorted form, much of value, is, nevertheless, as it stands unintelligible." (Dummett (1977), p. 250)

that *Logic is built into translation more fully than other systematic departments of science* (Quine (1986), p. 82). A difference in logic cannot, according to Quine, be really stated or communicated and is, therefore, to be seen as an illusion. A logician who endeavours to devise an alternative logic only changes the subject. When someone tries to, e.g. deny that in all cases everything follows from the conjunction of a statement and its negation, we have no reason to see him as really speaking about conjunction and negation[18]. Quine sums up his position in the famous dictum about the deviant logician who, *when [he] tries to deny the doctrine, he merely changes the subject* (Quine (1986), p. 81).

By accepting Quine's viewpoint, one gets to see logic as a prison we cannot escape. We simply cannot help using the logic that we in fact use. The Quinean arguments were criticized strongly by various authors, e.g. by Dummett. I think it is a pity that Quine did not speak much about whether different people or perhaps different cultures can in fact adhere to different logics and how the people who in some (not entirely clear but, for Quine, necessary) sense *have* a different logic can communicate about it. Yet, though Quine does not speak at much length about this, we can understand from his translation argument that they could never communicate their differences. Quine models the situation after that of radical translation, which is probably the source of the problem that we are not sure whether people can adhere to different logics and whether they could somehow explain their logical idiosyncrasies to each other. The route from the perspective of radical translation to that of disputes about various logics is probably not so straightforward. One has the feeling that, should the dialogue run the way Quine envisages it, something would be left uncommunicated[19]. *Prima facie* there is a real dispute between the adherents of different logics and this seems to show that Quine does not apply his maxim *save the obvious* correctly in the case of disputes between adherents of different logics. Apparently, the laws that the logicians are in dispute about are not obvious enough, not in the sense to be something we cannot abandon if we do not want to cease speaking intelligibly and understanding the others.

It nevertheless remains unclear what exactly Quine wants to say. Perhaps his message is that we all abide by classical logic (which he always defended strongly), even though we might be confused enough to think otherwise. Or maybe he admits that we can in fact adhere to different logic, yet these differences among us cannot be effectively stated and thus can be explained (away) as a sort psychological idiosyncrasy an individual or community might have. It should be fairly obvious that Quine's presentation gives a very uncharitable picture of the discussion between the adherents to different logics. Despite the intricacies involved, it seems clear that both the intuitionistic as well as the classical logician both speak of the concepts of disjunction and negation when they are in a dispute about the validity of the principle of the excluded middle (I will try to explain later what these underlying concepts consist of). Lots of authors have therefore accused Quine of contradicting his own positions from Quine (1951). Wasn't a vital part of his holism an attack on the synthetic/analytic distinction to show that every dispute can, in a way, be

18 Quine (1986), p. 81: "They think they are talking about negation, ¬, *not*; but surely the notation ceased to be recognizable as negation when they attempted at recognizing some conjunctions of the form $p.\neg p$ as true and stopped regarding such sentences as implying all others."

19 Quine would most likely think of this residuum as something idle like the Wittgensteinian beetle (see the paragraph 293 in Wittgenstein (2001)).

seen as a dispute about meanings as well as a dispute about matters of fact? Should the involvement of meaning thus be a reason for denying a debate its relevance (because the debate is then *merely about meanings and not about matters of fact*), then not just the debates about logical laws, but basically any rivalry between different scientific theories could be shown to be in this sense *idle*, which is surely an unacceptable position.

2.1 Choosing the best

I took Quine as an important example of an author arguing for the impossibility and unintelligibility of logical pluralism. As his argument is challenging and interesting, it also represents a defense of logical monism at what might be its best. I thus hope that we have at least sketched some arguments in favor of the thesis that there can indeed be more logics with different verdicts regarding the validity of some logical laws by showing the flaws of his reasoning. Yet opening the possibility of there being more correct logics clearly does not imply that logic is fundamentally arbitrary and the shape of, e.g. classical logic is as good as that of any systems of rules regarding logical constants that one might contrive. As in the case of geometry, there are very good reasons why the standard logic is the way it is and why it is accepted as such. To deny this means to subscribe to some sort of radical formalism, i.e. to the thesis that we have no understanding of logic independent of the specific mathematical theories (as say the axiomatization of classical logic) and that these theories by the same token do not relate to anything independent of them. Indeed, we can see such a tendency in Hilbert's responses to Frege in their correspondence about the status of geometry and possibilities of pluralism.[20] To suggest that anything, provided it is a consistent theory, can work as geometry (or, for that matter, logic) makes the very notion of geometry unintelligible. Yet, as Jaroslav Peregrin remarks (see Peregrin (2000)), Hilbert, as well as Frege, suffered rather from tendencies to overemphasize some relevant aspects of pluralism rather than from having incompatible, extreme positions.

While not surrendering to this kind of formalism, we can thus go on talking about a more moderate and reasonable version of logical pluralism as about a meaningful position and the rivalry of logics as a source of a reasonable kind of dispute. Thus obviously someone can adhere to classical logic while another logician can adhere to intuitionistic logic and so on. Nevertheless, as we said, logic belongs to the very fundamental layers of our conceptual schemes. Radical opposition to, say, classical logic (in the sense of asserting that completely different logical laws are valid than classical ones, i.e. not that just one or a few of the classical ones are problematic as, for example, the inuitionists claimed) can hardly be reasonably defended[21]. History shows us how harsh the classical/intuitionist dispute was, especially between Hilbert and Brouwer[22]. How have the logicians arrived at the peaceful coexistence we witness today? Have they given up the difficult debates about the nature of logic and its fundamental principles (as Dummett claims)?

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20 For a helpful overview of their debate, see Shapiro (1996)

21 Yet, of course, the dispute between adherents of different logics can very much resemble what Wittgenstein describes in paragraph 611 of Wittgenstein (1984): "Wo sich wirklich zwei Prinzipien treffen, die sich nicht miteinander aussöhnen, da erklärt jeder den Andern für einen Narren und Ketzer."

22 A nice oveview of this fight which involved not just arguments but also the use of power and coertion, can be found in Zach (2006).

Let us reflect on what is problematic in Quine's position once again. We can thereby hope to understand what makes us doubt the possibility of plurality among logicians. To begin with, Quine believes everyone is closed in one's own logic, i.e. that according to which one judges. Logic thus becomes, in some sense, a part of our nature which can be spelled out in two ways. The first will be discussed presently, while the other will come out as a result of our reflections at the end of the paper. The first and less fortunate was the one Quine inclined to; namely, the more literal. That is, logic is a part of our nature simply in the sense that we have the propensity to judge in accordance with it (for an illuminating discussion of this view, see chapter one of Peregrin (2014)). In this sense, there could indeed be a rivalry of logics which could be decided to some degree empirically. Undoubtedly, the shape of our logic has to correspond in some way to how we use the *logical words* in our natural languages, that is words such as *therefore*, *not*, *or*, etc. Quinean holism also shows us that such a fundamental discipline as logic has to react to empirical findings. Indeed, some authors went very far down this path, notably Putnam who in Putnam (1968) argues in favour of quantum logic as a theory vindicated by empirical data. Yet still, holism also teaches us that the contact of logic with experience should not be very intense, the gains of choosing the path of empirization of geometry would hardly prevail over all the disadvantages and confusion engendered thereby. The holist picture actually does not serve to deny that disciplines such as logic are *a priori*, despite appearances. Understood in a correct way, it rather shows us exactly why they are so immune to revision, though an important point is also that no such immunity is absolute. Important as it is, this possibility of changing logic should not be overemphasized, as Quine himself might have had the tendency to do in *Two dogmas*.

What should restrain us from an excessive amount of iconoclasm and empiricism with regard to logic is that we ought to care for the intelligibility and purity of the very notion of logic. Why should we, after all, call something directly testable by linguistic empirical findings *logic*? Not that we are not free to do so, the question is rather whether we would thus not lose something deeper and more interesting. Indeed, succumbing to the empiricist suggestions means forgetting the anti-psychologistic lessons taught by, among others, Kant, Husserl and Frege. The last author surely was not just stubborn when he insisted that logic has nothing to do with psychology. Perhaps, taking in the holistic lesson, we should say that his strict division of logic and psychology is too radical, yet basically it points in a correct direction. Departing from the Quinean position that logic is something we just have the tendency to comply with, we should say that the necessity of logic is normative in the sense that we need it in order to have the status of a rational human being, capable of reasoning and argumentation. This is the sense in which we want to see logic as a part of human nature.

2.2 Appreciating the normativist lessons

Can we, then, say that one of the logics does what it should do and is thus closer to its true purpose than the other ones? Clearly, should logic for example lack the rule of *modus ponens*, we would probably be at loss as to why to regard it as logic at all (though even logics abandoning this law have been proposed). Yet, the situation is different with respect to other contentious laws. Can the law of the excluded middle fail to hold in some contexts? There seems to be no definitive answer. Perhaps it does not really belong to the very notion of logic to have or not to have such a law? Logic appears to be fulfilling a function which can be fulfilled in various ways, i.e. by various different logics, as, for example, by those asserting the validity of the law of the excluded middle and those not doing so. What could such a function be? In fact, can we say that logic has got any concrete purpose, at all?

It is indeed difficult to say what the purpose of logic should be, while it is not so difficult to adduce arguments to the conclusion that it is actually quite idle. Indeed, history knows of figures who doubted the meaningfulness of logic, René Descartes, in particular, is famous for this (see paragraph 6 of chapter 2 of Descartes (1965)).

Logic seems to be saying nothing but what we already knew. In a different context, Kant said that logic always comes too late (a very helpful exposition of Kant's surprisingly modern views on logic can be found in Wolff (1995)). Kosta Došen shows in a very illustrative and systematic manner how the rules of logic always articulate only that which was already present in the structure of our discourse (Došen (1989)). We cannot ingore these insights, yet there is a possibility to accept them while retaining the conviction that logic is still a useful discipline. Jaroslav Peregrin draws (in Peregrin (2014)) a distinction between what he calls tactical and constitutive rules of a given game. While the tactical rules tell us how to play the game smartly, so that we can win or be otherwise successful, the constitutive rules enable this game to come into existence. The common mistake when speaking about the purpose of logic and the rules it provides us about, e.g. the way the conditional works, is to regard these as tactical rules. This means that we believe that the rules of logic help us have more true beliefs, be more successful at argumentation, etc. Such views then result in disappointments of the kind we just exemplified.

According to Peregrin, then, logic does not deal with tactical rules but rather with the constitutive rules of our language games. These spell out what makes the various rule-governed practices we engage in into language games. But how can this be, given that we already know what logic tells us, as is shown in Došen's article? Can't we see that language games work very well, even without logic? The point is, though, that by enabling these language games to come into existence in the first place, the inferential moves sanctioned by logic typically cannot be a reasonable part of these games, they are too basic in comparison with the other inferential rules. Let us compare the rules of logic with other rules which have a similar status. Wittgenstein discusses (in Wittgenstein (1984)), among many other examples of the things that we should be particularly certain about, the fact that everyone is sure about the name one has. Indeed, this is one of the constitutive rules of many of the rule-governed practices (language games) we engage in. Yet, of course, typically reminding someone that it is very important to know one's own name would be comical. It would be a piece of advice given desperately late to be of any use. Similarly, should we try to advise somebody to augment the list of his beliefs by $A \rightarrow B$ whenever he knows that he can infer A from B, the reaction would be most likely to be that of puzzlement or amusement.

2.2.1 What is the purpose of logic, then?

The doctrine we are advocating here is that of logical expressivism that was introduced by Brandom in his *Making it explicit* (Brandom (1994)). Logic, according to this view, serves to make explicit the inferential rules that are implicit in our language. The usefulness of this explicitness is not hard to see, as we are thus enabled to reflect on these rules and perhaps also change them. Logic is, as Brandom puts it, an organ of our *semantic consciousness*. In order to make this approach viable we, together with Brandom, have to admit that an inference can be legitimate without being sanctioned by logic, which means without being in a form of a logically valid inference rule. An example of such an inference would be

$$\frac{\text{Rex is a dog}}{\text{Rex is a mammal}}$$

Indeed, this is a correct inference step, not an abbreviation of one, no enthymeme resulting from omitting a premise such as *All dogs are mammals*. Indeed, to formulate this sentence we would need to already use logical tools, such as the general quantifier and conditional[23]. This sentence states the rule we implicitly follow in our language as its competent users.

Denying the legitimacy of such inferential steps not sanctioned directly by logic invites the question as to which of the many logics we should consider as underlying and legitimizing all the inferences we do. The position that every correct instance of inference is such because it complies to logic (called by Brandom *formalism* in Brandom (2000)) makes us wonder which logic it should be. The fact that there are apparently no viable ways of arguing in favour of any of these logics seems to me to be a strong reason against this formalism and for actually embracing the Brandomian stance of acknowledging extra-logical inferential correctness. And when we accept the inferences of this kind as correct, it is then meaningful to speak of expressing rules which found their correctness by the means of logic (instead of their being validated by logic in the first place). We can thus say that logic is capable of making the rules of inference explicit.

But still, can we say that logic is being done with one certain purpose, can we sum up all the logics we have under the heading of any purpose at all? And if the answer is yes, how can we be sure that we have found the true purpose and thus are trying to decide which logic is more logical than others by correct criteria? Obviously, the logicians who create new logics do not first ask themselves whether the new logic can in fact fulfill some particular general purpose, such as making inferential rules explicit. Yet, Brandom identified how logic and in particular the logical expressions such as *if, then, therefore, it is not true that*, etc. work in natural language. He described something like a natural logic. Yet, it is hardly thinkable that any of the mathematical logics could behave just like the natural logic we use. In fact, speaking of one natural logic is an abstraction because every natural language together with its logical vocabulary is a flux and the rules of it are implicit and unstable. The rules of logic, on the other hand, are very precise and articulated.

[23] We would obviously formalize this sentence as $\forall x(Dog(x) \rightarrow Mammal(x))$.

Thus, there is a fundamental discrepancy with the original purpose of logic and the logical systems we know so that it makes no sense to say that one of them is some sense fulfilling the purpose of logic better than the others. The logical systems form a scientific, systematic prolongation of the natural logic which serves to express the inferential relationships which are the very core of our language games, or at least those fundamental ones which include giving and asking for reasons. As prolongations they are fundamentally dependent on this original activity of ours, they cannot begin anew in some pure or perfect manner.

As we know from Quine, "science is a continuation of common sense" (Quine (1951)). The language of science does not work in a fundamentally different way than the natural language, yet it puts special emphasis on exactness. Thus, it not only creates its specific terminology but also can make very good use of a stricter logic. Yet the study of how the logic of science and particularly of mathematics[24] could serve as the tool of expression of inferential rules in science led to the construction of different systems which can all plausibly fulfill the expressivist role. Here it is good to note that the term *implicit inferential rule* has to be taken with a grain of salt. One of the features of an implicit inferential rule is its high degree of indefiniteness. It often bears the potential to be converted into various different explicit rules which contradict each other. The explicit statement of the rule is thus an act of creation, as well. We do not just retell explicitly what was already present implicitly, we reshape it as well. This means that there are indeed more possibilities as to which rules to state and accept explicitly and still in fact see them as a continuation of the practices guided by the implicit rules that existed before.

2.3 Two freedoms of choice

Calling the approach to logic proposed by Brandom *logical expressivism* is basically right and suggestive but, we have to remember that we are not merely expressing something, which is simply already there, we are also thereby giving it a shape. There is this fundamental freedom to formulate various inferential rules on the basis of the workings of the already functioning discourse. To be sure, this is by no means an anarchy, the discursive practices determine what may be regarded as an expression of the rules which govern them to a great degree but not absolutely. I believe expression is a much bigger part of the enterprise than stipulation and therefore it is basically correct to speak of logical expressivism.

Nevertheless, the creative element present in formulating the rules of inference is very important and we have seen two kinds of the (very restricted) freedom we have when we formulate inference rules. Firstly, we can choose the exact shape of the rules and thus the logical relationships between propositions and thus between concepts. For example, consider the following potential law of inference in biology:

$$\frac{\text{XYZ is a dog}}{\text{XYZ has got lungs}}$$

[24] There is no reason to think that all the sciences share the same needs, not all of them are served by logic as much as mathematics.

It appears to be indeterminate whether we should regard this rule as valid or otherwise. Certainly the ordinary talk about dogs and possibly also the discourse of scientific zoology leaves this under-determined and we can thus choose. If we encountered animals which shared with dogs as we know them all their properties, except that they lacked lungs, we would hardly be sure whether we should consider them as a specific kind of dog or rather as a different category of animal. Of course, we do not have to do this if this particular decision is not of great importance to us. As unimportant as this individual decision may be, language as we know it is formed by our capacity to make lots of such decisions and our activity of formulating rules thus involves the kind of freedom we are talking about. The first freedom thus consists in our (limited) leeway in specifying which sentences are in logical relationships. Here, we just saw an example of two sentences which were indeterminate as to whether the first entailed the other one.

It is nevertheless the other kind of freedom that is important for us now, namely our freedom to choose the tools of this expression and formulation of inferential rules, i.e. our logic. In the nineteenth and twentieth century, mathematical methods were developed which enabled us to construct various logics. Every such logic has got many properties, some of which are postulated and some of which discovered by the mathematical logicians. Because of differences in these properties (such as compactness, completeness, decidability, etc.), the various logics can be seen as different kinds of an instrument of expressing the inferential rules, i.e. of *making them explicit*. Just as we can have many different hammers which can be useful for different purposes, so we can have various logics that express the inferential rules in our language games. It is true that by constructing new kinds of hammers we can also use them for quite novel purposes, thus partly changing our original conceptions of what a hammer is for. Thus, when logicians develop new logics, we can get new ideas about what logic can be useful for. Yet, just as at a certain point it is not meaningful to call something a hammer because it is perhaps too big, so it might be unreasonable to call certain mathematical theories *logics* because they cannot be reasonably seen to serve the purpose of expressing inferential relationships, even though they contain elements which behave similarly as, say, the connectives of classical logic. Nevertheless, the development of new logics as well as further mathematical discoveries about those we already know can bring us to a new understanding of what it means to make the rules guiding discursive practices explicit.

Summing up my position, I claim that there is a mutual influence between the philosophical reflection of logic and its purpose and the mathematical study and creation of new systems thereof. This mutual influence is a motor of development of logic as a discipline in general. Despite this dynamic relationship, there are still boundaries as to what can reasonably be regarded as logic, namely a system which can make explicit the inferential rules in our language games.

3. Back to geometry

Though our primary focus is on the plurality of logics, we also discussed the plurality of geometries, hoping to find useful lessons about logic therein. So far, we have arrived at logical expressivism as a philosophical stance which enables us to explain the plurality of logics thanks to its analysis of what a logic should actually do. We saw that logical expressivism is built upon the notion of inference, which is correct without being sanctioned by logic. In other words – material, as opposed to formal, inference. Acceptance of this notion can, on the other hand, also be made plausible by the plurality of logics. Logical pluralism and logical expressvism thus support each other. In the case of geometry we arrived primarily at holism and thereby saw that various geometries can be a part of broader theories helping us deal with the world. Limited holism is also a reasonable position for the philosophy of logic. The relationship between logic and mathematics can, particularly, be shaped in various ways, giving us leeway to choose whether to make adjustments rather in logic or in mathematics and in particular decide how much mathematics should be a part of logic. Yet, this holism should respect the expressivist role of logic in order to still be able to use the word *logic* meaningfully at all.

Now the holism we proposed for geometry was left rather abstract and general. This means that it does not provide much help for understanding what the specific task of this discipline is as opposed to, e.g. mechanics which interrelates closely with it. Consequently, we have so far arrived at no demarcation or at least guidelines for drawing a demarcation between what is and what is not a geometry. Nevertheless, Euclidian geometry truly deserves to be called geometry if anything does. The hyperbolic and elliptic geometries were plausibly shown to deserve the same, yet where is the boundary, how much can we change these theories and still talk about alternative geometries? As in the case of logic, we will not give any concrete list of possible shapes geometry can take, i.e. a list of acceptable theories. Yet, logical expressivism was found to be a fruitful formulation of what criterion we should use when deciding what is and what is not a logic, namely whether it could serve to make inference rules explicit.

As we tried to further address the problem of logical pluralism by revisiting that of geometrical pluralism, let us now try to apply what we learned about logic to geometry as well. Let us try to push the analogy as far as possible and speak of geometrical expressivism. What would such a doctrine involve? And does it make sense in the first place? Recall that there is a tradition of metaphorical talk of logical space, as can be seen, e.g. in Wittgenstein (2003). One of the features of space is that it is necessary for the entities which inhabit it. Indeed, they cannot exist but inside it. It constitutes relations between these entities, such as being above one another, being on the right of one another, etc[25]. Similarly, in the logical space there are specific relationships such as being a consequence of, being incompatible with, being a conjunction of two different propositions, etc. And indeed these relationships are essential for the propositions to be propositions at all. To disambiguate a little, when we have three physical objects, say a chair, a table and a ball, then the ball does not have to be between the other two, the chair does not have to be in front of the table, the ball on the table, etc. Yet clearly they have to be capable of entering into such relationships in order to be material objects all. In the same manner, when we speak of three propositions A, B and C, then A clearly does not have to be a consequence of B, C, its negation or perhaps the conjunction of both B and C. Yet, they have to be capable of entering such relations, though this time – as opposed to the physical objects – not between each other but perhaps with different propositions altogether.

[25] These spatial relations make sense, of course, only from a particular spectator's perspective.

Thus necessity and essentiality are among the important features physical and logical space share. The other is that they are in fact implicit, in a way invisible. We clearly see just the physical objects in the space, not the space itself. The logical relations between propositions and indeed their respective positions in the logical space is something we also do not realize primarily and they indeed have to be made explicit if we want to reason about them. Making both the logical and physical space explicit is not always useful. Indeed, it is called for when we encounter problems with the relations we are aware of just implicitly, when we cannot just go on in our practices, but have to analyze the situation more theoretically (we mentioned the somewhat trivial example with dogs and lungs but better ones could easily be provided). The logical analysis is typically useful (and sometimes even necessary) when we are at a loss as to whether a certain argument should be regarded as valid. We already saw that this also involves some freedom on our side. The need for making the spatial relationships explicit arises typically when we want to prevent some unfortunate accidents such as collapse of a bridge or when we react to those accidents which have already happened. Here we can apparently also speak of the twofold freedom we discovered when reflecting on similar features of logic. Given a logic, we saw that we have freedom how to articulate the rules by its means. On the other hand, there is also some freedom in choosing the logic we use for the formulation.

Similarly, there is some freedom in how we articulate the spatial relations between objects, ending possibly in (at least slightly) incompatible formulations. More importantly for us, though, there is a freedom to choose different geometries for the purposes of this articulation. Just as the ordinary practices of giving and asking for reasons do not force on us whether some logical laws, such as the excluded middle, hold, so we can also say that the ordinary practice of articulating the spatial relationships leaves under-determined at least whether the parallel postulate holds. We need to make a decision concerning this and similar issues only when the natural practice, the *natural geometry*, does not suffice, e.g. when human lives might be put at risk when bridges are built or when we are otherwise extremely dependent on the exactness of our understanding of geometrical notions (as in astronomy where we have to compensate for our inability to physically approach the examined objects).

4. Summary

We have seen that both the problem of the plurality of geometries and that of plurality of logics can be understood by seeing their role as that of expressing implicit spatial and logical relationships. In fact, we did not arrive at a specific demarcation either of logic or of geometry, but tried to understand what the nature of those disciplines is in the light of the plurality which cannot be denied. Considering the relation, or rather the gap, between the mathematical systems and the practices which underlie them, we see that the attempts at demarcation have indeed little hope of helping us understand geometry or logic much more deeply.

Acknowledgments

This article was supported by grant 17-15645S *Logical models of reasoning and argumentation in natural language*, led by Jaroslav Peregrin from Czech Academy of Sciences.

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PHILOSOPHICA ET HISTORICA 2 / MISCELLANEA LOGICA

PRESERVING MEASURABILITY WITH COHEN ITERATIONS

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ABSTRACT
We describe a weak version of Laver indestructibility for a μ-tall cardinal κ, $\mu > \kappa^+$, where "weaker" means that the indestructibility refers only to the Cohen forcing at κ of a certain length. A special case of this construction is: if μ is equal to κ^{+n} for some $1 < n < \omega$, then one can get a model V^* where κ is measurable, and its measurability is indestructible by $\mathrm{Add}(\kappa, \alpha)$ for any $0 \leq \alpha \leq \kappa^{+n}$ (Theorem 3.3).
Keywords: Cohen forcing, measurability
AMS subject code classification: 03E35, 03E55

1. Introduction

Assume κ is supercompact. In [7], Laver defined an iteration P of length κ such that in $V[P]$,[1] κ is still supercompact and every further κ-directed closed forcing preserves the supercompactness of κ (P is often called the *Laver preparation*). We also say that κ is Laver-indestructible in $V[P]$. The proof of this indestructibility result is essentially based on two useful properties of a supercompact cardinal κ in V: (i) for every $\mu \geq \kappa$, one can choose an elementary embedding $j : V \to M$ with critical point κ such that M is closed under μ-sequences existing in V; this closure is then used to find a *master condition* in M and proceed with a lifting argument which ensures that supercompactness is preserved,[2] (ii) there is a single function $f : \kappa \to V_\kappa$ such that for every $x \in V$, one can choose an embedding j in (i) so that $j(f)(\kappa) = x$ (this f is often called the *Laver function*).

A typical example of a κ-directed closed forcing is the Cohen forcing at κ, which we will denote by $\mathrm{Add}(\kappa, \alpha)$,[3] where α is any ordinal larger than 0. The fact that over $V[P]$, $\mathrm{Add}(\kappa, \alpha)$ preserves the measurability of κ is very useful when one wishes to use some

1 $V[P]$ indicates a P-generic extension of V whenever it is not important to distinguish specific P-generic filters. For instance the statement "φ holds in $V[P]$" means that φ holds in $V[G]$ for every P-generic filter G.

2 Assume $j : V \to M$ is an elementary embedding, P is a forcing notion, G is P-generic over V, and H is $j(P)$-generic over M. Then a sufficient condition for j to *lift*, i.e. a sufficient condition for the existence of $j^+ : V[G] \to M[H]$ with $j^+ \restriction V = j$, is that we have $j''G \subseteq H$. With supercompactness, we can often argue that $j''G$ is a condition in M (a master condition), and H can then be built below this master condition. For more details, see [3].

3 Formally speaking, conditions in $\mathrm{Add}(\kappa, \alpha)$ are partial functions of size $< \kappa$ from $\kappa \times \alpha$ to 2. The ordering is by reverse inclusion.

https://doi.org/10.14712/24647055.2017.13

large cardinal properties of κ in $V[P][\mathrm{Add}(\kappa,\alpha)]$ (see for instance [4] where a model with the tree property at κ^{++}, κ strong limit singular with cofinality ω, is constructed starting with a supercompact κ).

A natural question is whether a "Laver-like" indestructibility is available also for smaller large cardinals. As it turns out, it is the property (i) above which is more important: it is known that for instance a strong cardinal[4] κ has the analogue of the Laver function, but it is not known whether it can be made indestructible under κ-directed closed forcings.[5]

In this short paper we use the idea of Woodin (as described in [2]) to argue that it is possible to have a limited indestructibility of a μ-tall cardinal[6] κ, $\kappa^+ < \mu$ regular, in the sense that we can successively extend $V \subseteq V^1 \subseteq V^*$ so that forcing with $\mathrm{Add}(\kappa,\mu)$ over V^* yields the measurability of κ. See Section 2.

If $\mu = \kappa^{+n}$, $1 < n < \omega$, we can say more. If κ is $H(\kappa^{+n})$-hypermeasurable[7], V^* has the property that forcing with $\mathrm{Add}(\kappa,\alpha)$ over V^* for $0 < \alpha \leq \kappa^{+n}$ yields the measurability, in fact hypermeasurability, of κ (Theorem 3.1 and Theorem 3.3). Note that in V^*, κ may actually stop being measurable[8] depending on the iteration P_κ which gives $V^* = V^1[P_\kappa]$; compare the constructions in Theorem 3.1 and 3.3.

Remark 1.1. We assume that the reader is familiar with the lifting arguments. The general reference is [3]; the more specific constructions used in the present paper are also given in [2].

2. Tall cardinals

In this section, we assume GCH. Let κ be μ-tall cardinal for some regular $\kappa^+ < \mu$.

Let $j : V \to M$ be a μ-tall embedding with the extender representation:

$$M = \{j(f)(\alpha) \mid f : \kappa \to V \ \&\ \alpha < \mu\}.$$

In particular, M is closed under κ-sequences in V and $\mu < j(\kappa) < \mu^+$. Let U be the normal measure derived from j, and let $i : V \to N$ be the ultrapower embedding generated by U. Let $k : N \to M$ be elementary so that $j = k \circ i$. Note that κ is the critical point of j, i and j, i have support κ, i.e. every element of M and N is of the form $j(f)(\alpha)$, or $i(f)(\kappa)$ respectively, for some f with domain κ. In contrast, the critical point of k is $(\kappa^{++})^N$ and k has support which we denote ν, where $(\kappa^{++})^N < \nu < i(\kappa)$, i.e. every element of M can be written as $k(f)(\alpha)$ for some f in N with domain ν.[9]

Let P denote the forcing $\mathrm{Add}(\kappa,\mu)$ in V, $Q = i(P)$, and let g be a Q-generic filter over V. Then the following hold:

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4 A regular cardinal κ is *strong* if for every $\mu \geq \kappa$ there is $j : V \to M$ with critical point κ and $H(\mu) \subseteq M$.

5 A non-supercompact strong cardinal κ can be indestructible under κ-directed closed forcings by a method of [1], but κ needs to be supercompact in the ground model.

6 There is $j : V \to M$ with critical point κ such that M is closed under κ-sequences and $j(\kappa) > \mu$.

7 κ is *$H(\mu)$-hypermeasurable* (also *$H(\mu)$-strong*) if there is an elementary embedding $j : V \to M$ with critical point κ such that $j(\kappa) > \mu$, $H(\mu) \subseteq M$, and M is closed under κ-sequences in V.

8 If in V^*, κ is not measurable, and it is measurable again in $V^*[\mathrm{Add}(\kappa,\alpha)]$ (for a specific α), it is more appropriate to call this step a "resurrection" of the measurability of κ.

9 ν needs to have the property that $k(\nu) \geq \mu$; some such ν always exists.

Theorem 2.1. *GCH. Forcing with Q preserves cofinalities and the following hold in $V[g]$:*

(i) j lifts to $j^1 : V[g] \to M[j^1(g)]$, where j^1 restricted to V is the original j.

(ii) i lifts to $i^1 : V[g] \to N[i^1(g)]$, where i^1 restricted to V is the original i. $N[i^1(g)]$ is the measure ultrapower obtained from j^1.

(iii) k lifts to $k^1 : N[i^1(g)] \to M[j^1(g)]$, where k^1 restricted to N is the original k.

(iv) g is Q-generic over $N[i^1(g)]$.

Proof. We show that Q is κ^+-closed and κ^{++}-cc in V. Closure is obvious by the fact that N is closed under κ-sequences in V. Regarding the chain condition, notice that every element of Q can be identified with the equivalence class of some function $f : \kappa \to \mathrm{Add}(\kappa, \mu)$. For $f, g : \kappa \to \mathrm{Add}(\kappa, \mu)$, set $f \le g$ if for all $i < \kappa$, $f(i) \le g(i)$; it suffices to check that the ordering $\le$ on these f's is κ^{++}-cc. Let A be a maximal antichain in this ordering; take an elementary substructure $\bar{M}$ in some large enough $H(\theta)$ of V which contains all relevant data, has size κ^+ and is closed under κ-sequences. Then it is not hard to check that $A \cap \bar{M}$ is maximal in the ordering (and so $A \subseteq \bar{M}$), and therefore has size at most κ^+.

(i) and (ii). These follow by κ^+-distributivity of Q in V and the fact that j, i have support κ: the pointwise image of g generates a generic for $j(Q)$ and $i(Q)$, respectively.

(iii). $i(Q)$ is $i(\kappa^+)$-closed in N, and since $\nu < i(\kappa^+)$, we use the distributivity of $i(Q)$ and the fact that k has support ν to argue that the pointwise image of $i^1(g)$ generates a generic filter which is equal to $j^1(g)$ by commutativity of j, i, k.

(iv). Q is $i(\kappa^+)$-cc in N and $i(Q)$ is $i(\kappa^+)$-closed in N. There are therefore mutually generic over N by Easton's lemma. □

Remark 2.2. It would be tempting to expect that j^1 is still $H(\mu)$-hypermeasurable if the original j was: however g is not included in $M[j^1(g)]$ and j^1 is therefore just μ-tall. There are some delicate issues involved if one wishes to preserve the $H(\mu)$-hypermeasurability of κ in Theorem 2.1. A natural strategy is to prepare below κ by a reverse Easton iteration. This approach is taken in [2] where it is also shown that if $\mu = \kappa^{++}$, then Q is isomorphic to $\mathrm{Add}(\kappa^+, \kappa^{++})$ and thus the preparation can be implemented by iterating $\mathrm{Add}(\alpha^+, \alpha^{++})$ at all inaccessible $\alpha \le \kappa$. In [5], this representation is shown for $\mu = \kappa^{+n}$ for $2 \le n < \omega$, i.e. $i(\mathrm{Add}(\kappa, \kappa^{+n}))$ is isomorphic to $\mathrm{Add}(\kappa^+, \kappa^{+n})$. It seems it is possible to continue up to the first cardinal above κ with cofinality κ, but it is unclear whether it can be extended further.

Remark 2.3. The loss of the $H(\mu)$-hypermeasurability of j^1 may prevent the use of this method in more complicated situations (such as a subsequent definition of Radin forcing to achieve results of a more global character).

Let us work in the model $V[g] = V^1$ and let us use the notation $j^1, i^1, k^1, V^1, M^1, N^1$ to denote the resulting models and embeddings in Theorem 2.1. Using a fast-function forcing of Woodin, we can assume that there is $f : \kappa \to \kappa$ in V such that $j(f)(\kappa) = \mu$. Let us denote $f(\alpha)$ by μ_α; let $C(f)$ denote the closed unbounded set of the closure points of f: if $\alpha \in C(f)$, then for all $\beta < \alpha, f(\beta) < \alpha$.

Theorem 2.4. *There is a forcing iteration R_κ defined in V^1 such that*

$$V^1[R_\kappa][\mathrm{Add}(\kappa, \mu)] \vDash \kappa \textit{ is } \mu\textit{-tall},$$

where $\mathrm{Add}(\kappa,\mu)$ *is defined in* $V[R_\kappa]$.

Proof. Define R_κ to be the following Easton-supported iteration:

$$R_\kappa = \langle (R_\alpha, \dot{Q}_\alpha) \mid \alpha \in C(f), \alpha \text{ inaccessible}\rangle, \tag{2.1}$$

where $\dot{Q}_\alpha$ denotes the forcing $\mathrm{Add}(\alpha,\mu_\alpha)$.

The proof uses the usual surgery argument (see [3]) with Fact 2.5 which allows us to use the generic filter g added in V^1 (for the i^1-image of $\mathrm{Add}(\kappa,\mu)^{V^1}$) in the model $V^1[R_\kappa]$ (for the proof, see Fact 2 in [2]).[10]

Fact 2.5. *Let S be a κ-cc forcing notion of cardinality κ, $\kappa^{<\kappa} = \kappa$. Then for any μ, the term forcing $Q_\mu = \mathrm{Add}(\kappa,\mu)^{V[S]}/S$ is isomorphic to* $\mathrm{Add}(\kappa,\mu)$.

Now we proceed with the proof of Theorem 2.4. Let $G_\kappa * H$ be $R_\kappa * \mathrm{Add}(\kappa,\mu)^{V^1[R_\kappa]}$-generic over V^1. Using the standard methods, lift[11] in $V^1[G_\kappa * H]$ the embeddings j^1, i^1, k^1 to R_κ, obtaining commutative triangle $j^1 : V^1[G_\kappa] \to M^1[j^1(G_\kappa)]$, $i^1 : V^1[G_\kappa] \to N^1[i^1(G_\kappa)]$, and $k^1 : N^1[i^1(G_\kappa)] \to M^1[j^1(G_\kappa)]$.

Using the elementarity of i^1, Fact 2.5 applied with $S = i^1(R_\kappa)$ and $i^1(\mathrm{Add}(\kappa,\mu))$ shows that g – which is present in V^1 – yields a generic filter g' for the forcing $i^1(\mathrm{Add}(\kappa,\mu))$ of $N^1[i^1(G_\kappa)]$. The pointwise image of g' via k^1 generates a $j^1(\mathrm{Add}(\kappa,\mu))$-generic filter over $M^1[j^1(G_\kappa)]$, which is then modified by the standard surgery argument to allow for lifting j^1 to $V^1[G_\kappa * H]$ (for details see [2]); i.e. if we denote the lifting of j^1 by j^2, then

$$j^2 : V^1[G_\kappa][H] \to M^1[j^1(G_\kappa * H)]$$

witnesses the measurability, and in fact μ-tallness, of κ. □

3. Hypermeasurable cardinals

It seems natural to extend Theorem 2.4 and have that the measurability of κ ensured by $\mathrm{Add}(\kappa,\alpha)$ for any ordinal α, $0 < \alpha \le \mu$. We will show that this can be achieved with some additional assumptions on μ. For concreteness, we will focus on the example where $\mu = \kappa^{+n}$ for some $1 < n < \omega$.

First, in Theorem 3.1, we provide a standard construction which actually forces κ to stop being measurable in V^*; the measurability of κ is then resurrected by $\mathrm{Add}(\kappa,\alpha)$ for any $\kappa^+ \le \alpha \le \kappa^{+n}$.

Theorem 3.1. *(GCH) Let $1 < n < \omega$ be fixed and assume κ is $H(\kappa^{+n})$-hypermeasurable. Then there is an iteration P^1 such that in $V[P^1] = V^1$, κ is still κ^{+n}-hypermeasurable, and for some reverse Easton iteration P_κ defined in V^1, κ stops being measurable in $V^* = V^1[P_\kappa]$. In V^*, the measurability – in fact the hypermeasurability – of κ is resurrected by Cohen forcing $\mathrm{Add}(\kappa,\alpha)$ for any $\kappa^+ \le \alpha \le \kappa^{+n}$.*

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10 Recall that Q_μ – mentioned in Fact 2.5 – is the term forcing defined as follows: the elements of Q_μ are names τ such that τ is an S-name and it is forced by 1_S to be in $\mathrm{Add}(\kappa,\mu)$ of $V[S]$. The ordering is $\tau \le \sigma \leftrightarrow 1_S \Vdash \tau \le \sigma$.

11 For simplicity, we use the notation j^1, i^1, k^1 to denote the partial liftings of the embeddings j^1, i^1, k^1.

Proof. Let j be an extender embedding witnessing the $H(\kappa^{+n})$-hyper-measurability of κ, and let i be a normal embedding generated by the normal measure U derived from j. Recall Lemma 3.2 from [5] which implies that if $i : V \to N$ is an embedding generated by a normal measure on κ, then

$$\text{Add}(i(\kappa), i(\kappa)^{+n})^N \cong \text{Add}(\kappa^+, \kappa^{+n}). \tag{3.2}$$

Define P^1 is an Easton-supported iteration

$$\langle (P^1_\alpha, \dot{Q}_\alpha) \mid \alpha < \kappa, \alpha \text{ is inaccessible} \rangle * \dot{Q}_\kappa,$$

where for an inaccessible $\beta \leq \kappa$, $\dot{Q}_\beta$ is $\text{Add}(\beta^+, \beta^{+n})$ of $V[P^1_\beta]$.

Let $G_\kappa * g$ be $P^1_\kappa * \dot{Q}_\kappa$-generic over V, and denote $V[G_\kappa * g]$ by V^1. Let j^1 and i^1 be the liftings of j and i.

In V^1 define P_κ as an Easton supported iteration:

$$P_\kappa = \langle (P_\alpha, \dot{Q}_\alpha) \mid \alpha < \kappa \text{ is inaccessible} \rangle, \tag{3.3}$$

where $\dot{Q}_\alpha$ denotes the forcing $\text{Add}(\alpha, \alpha^{+n})$ of $V^1[P_\alpha]$.

First note that κ stops being measurable in $V^* = V^1[P_\kappa]$ by the application of the gap-forcing theorem of [6]: a hypothetical embedding k with critical point κ found in V^* could be written as an embedding from $V^1[P_\kappa]$ to some $N[j(P_\kappa)]$, with $N \subseteq V^1$; in particular a generic filter for $j(P_\kappa)$ would need to add a non-trivial generic filter at stage κ which cannot be found in $V^1[P_\kappa]$.

The rest of the Theorem follows from the following Claim:

Claim 3.2. *Let α be an ordinal, $\kappa^+ \leq \alpha \leq \kappa^{+n}$. Then κ is still measurable in $V^1[P_\kappa]$ $[\text{Add}(\kappa, \alpha)]$, where $\text{Add}(\kappa, \alpha)$ is defined in $V^1[P_\kappa]$.*

Proof. It suffices to show the Claim for α's which are cardinals. So assume $\kappa^{+m} = |\alpha|$ for some $1 \leq m \leq n$. Choose in V^1 an embedding $j_m : V^1 \to M_m$ which witnesses the $H(\kappa^{+m})$-hypermeasurability of κ with $\kappa^{+m} < j_m(\kappa) < \kappa^{+m+1}$ (this is possible since $2^\kappa = \kappa^+$ in V^1). By the definition of P_κ, $j_m(P_\kappa)(\kappa)$ is equal to $\text{Add}(\kappa, \kappa^{+n})^{M_m[P_\kappa]}$. Since $(\kappa^{+n})^{M_m}$ has size κ^{+m} in V^1, $\text{Add}(\kappa, \kappa^{+m})^{V^1[P_\kappa]}$ is equivalent to $\text{Add}(\kappa, \kappa^{+n})^{M_m[P_\kappa]}$, and therefore the generic for $\text{Add}(\kappa, \kappa^{+m})^{V^1[P_\kappa]}$ provides a generic for $\text{Add}(\kappa, \kappa^{+n})^{M_m[P_\kappa]}$. The argument is then finished as in Theorem 2.4, using the fact that the generic g for $i^1(\text{Add}(\kappa, \kappa^{+n}))$ is also generic for $i^1(\text{Add}(\kappa, \kappa^{+m}))$. □

This concludes the proof of Theorem 3.1. □

Note that the method in the proof of Theorem 3.1 does not work for the case of α smaller than κ^+: every elementary embedding $k : V^1 \to M$ with critical point κ sends κ above κ^+ and therefore $\kappa^+ \leq |\kappa^{+n}|$ in V^1; thus $k(P_\kappa)(\kappa)$, which is $\text{Add}(\kappa, \kappa^{+n})^{M[P_\kappa]}$, is in $V^1[P_\kappa]$ equivalent to the Cohen forcing at κ of length at least κ^+. It follows that to lift the embedding, we need to force over $V^1[P_\kappa]$ with a Cohen forcing at κ of length at least κ^+. If $\alpha < \kappa^+$, this condition is not satisfied. We remedy this by a more complicated construction in Theorem 3.3.

Theorem 3.3. *With the assumptions and the notation as in Theorem 3.1, one can define P_κ so that κ is measurable in V^*, and its measurability – in fact hypermeasurability – is indestructible by $\text{Add}(\kappa, \alpha)$ for any $0 < \alpha \leq \kappa^{+n}$.*

Proof. Modify the definition of P_κ in (3.3) so that at an inaccessible $\alpha < \kappa$, $\dot{Q}_\alpha$ is chosen generically[12] amongst the following forcings: $\{1\}$ (the trivial forcing), and $\mathrm{Add}(\alpha, \alpha^{+k})$, for $0 \leq k \leq m$.

Then one can argue that κ is still measurable in V^*: while lifting the embedding j^1, it suffices to work below a condition in $j^1(P_\kappa)$ which chooses the trivial forcing $\{1\}$ at stage κ.

To argue that for any $0 < \alpha \leq \kappa^{+n}$, κ is still measurable in $V^*[\mathrm{Add}(\kappa, \alpha)]$, work below a condition in $j^1(P_\kappa)$ which chooses the right forcing at stage κ. □

4. Open questions

Q1. Is it possible to generalise Theorem 2.4 so that μ is still $H(\mu)$-hypermeasurable if the original embedding j was $H(\mu)$-hypermeasurable? This would require some sort of preparation below κ in the model V^1 (analogously to the methods in Theorem 3.1).

A related question is this:

Q2. Is it possible to characterise the forcings $i(\mathrm{Add}(\kappa, \mu))$, where $i : V \to N$ is a normal measure ultrapower as in Theorem 2.1? We know that this forcing does not collapse (it is κ^+-closed and κ^{++}-cc in V), but does it have a uniform representation? In particular, is it isomorphic to $\mathrm{Add}(\kappa^+, \mu)$ of V?

Acknowledgments

The work was mainly supported by FWF/GAČR grant I 1921-N25.

A support of travel grant Mobility 7AMB15AT035 is also acknowledged.

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12 The "lottery preparation" in the terminology of Hamkins.

2017 ACTA UNIVERSITATIS CAROLINAE PAG. 33–43
PHILOSOPHICA ET HISTORICA 2 / MISCELLANEA LOGICA

DIAGONAL ARGUMENTS

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ABSTRACT
It is a trivial fact that if we have a square table filled with numbers, we can always form a column which is not yet contained in the table. Despite its apparent triviality, this fact can lead us the most of the path-breaking results of logic in the second half of the nineteenth and the first half of the twentieth century. We explain how this fact can be used to show that there are more sequences of natural numbers than there are natural numbers, that there are more real numbers than natural numbers and that every set has more subsets than elements (all results due to Cantor); we indicate how this fact can be seen as underlying the celebrated Russell's paradox; and we show how it can be employed to expose the most fundamental result of mathematical logic of the twentieth century, Gödel's incompleteness theorem. Finally, we show how this fact yields the unsolvability of the halting problem for Turing machines.
Keywords: diagonalization, cardinality, Russell's paradox, incompleteness of arithmetic, halting problem

1. General formulation

Imagine a square table populated by natural numbers.

5	0	3
1	1	3
3	7	3

Is it possible to add a column that the table does not yet contain? There are, of course, many such columns that could be added. Now suppose that the table is populated only by zeros and ones.

0	0	1
1	1	0
1	0	0

https://doi.org/10.14712/24647055.2017.14

Is it still possible to add a new column? Of course, it is – for example 0, 0, 0; or 1, 1, 1. Now suppose that the table is very large. Can we still do the same? It seems that the answer must still be positive, though now it might be not so easy to find a new column. Here is an easy method: make the first number of the new column different from that in the first row of the first column of the original table, make the second number in the new column different from that in the second row of the second column, etc. Hence, the number in the *n*th row of the new column is different from that in the *n*th row of the *n*th column and the new column is thus different from each column of the original table.

0	0	1	**1**
1	1	0	**0**
1	0	0	**1**

We can call this method diagonalization, and we can call the column produced by diagonalization the *antidiagonal* of the table (we will abreviate it to AD). Note that we can speak about *the* antidiagonal only thanks to the fact that the table we talk about cannot contain more than two numbers (0 and 1 in our case). If the values in the table were allowed to be drawn from a set consisting of more than two elements, there would be many antidiagonals. Note also that the presupposition of the application of diagonalization is that the table is square, i.e. that the number of rows of the original table equals the number of its columns.

Simple as this method may seem to be, it lays the foundation of many path-breaking results of logic in the second half of the nineteenth and the first half of twentieth century.[1] Let us assume that each row in a table we are considering has a label and let us use the sign **D** for the set of all the labels. Each column of the table can then be considered as a function mapping **D** (in our introductory examples **D** could be {1,2,3}, for the tables have three rows) on a set **R** of those values that can occur within the table (in our first example, **R** may be {0,1,3,5,7} (or any set containing it), in the second one it would be {0,1}).

	f_1	f_2	f_3	f_4	f_5	...
d_1	$f_1(d_1)$	$f_2(d_1)$	$f_3(d_1)$	$f_4(d_1)$	$f_5(d_1)$	...
d_2	$f_1(d_2)$	$f_2(d_2)$	$f_3(d_2)$	$f_4(d_2)$	$f_5(d_2)$	...
d_3	$f_1(d_3)$	$f_2(d_3)$	$f_3(d_3)$	$f_4(d_3)$	$f_5(d_3)$	...
d_4	$f_1(d_4)$	$f_2(d_4)$	$f_3(d_4)$	$f_4(d_4)$	$f_5(d_4)$	...
d_5	$f_1(d_5)$	$f_2(d_5)$	$f_3(d_5)$	$f_4(d_5)$	$f_5(d_5)$	...
...	...	...	...	...	...	...

1 In this paper we concentrate at the most perspicuous presentation of the diagonal argument. For more detailed, deeper and more technical accounts see Smullyan [1994], Boolos et al. [2002], or Gaifman [2006].

The table thus presents a set **F** of functions from **D** to **R**, such that the number of elements of **F** coincides with that of **D** (the table is square); and the diagonal method shows that there is a function from **D** to **R** which does not belong to **F:**

Theorem. Let **F** be a set of functions with the domain **D** and range **R**. Let **R** consist of at least two elements. Then, if the cardinality of **F** is the same as that of **D**, there exists a function from **D** to **R** which is not an element of **F.**
Proof: Let i be a one-one mapping of **D** on **F**. Let **f** be such that $\mathbf{f}(x) \neq f_x(x)$, where $f_x = i(x)$, for every x from **D**. Then **f** is – obviously – not an element of **F**.

This formulation allows us to extend our considerations to infinite “tables” – even to “tables” with a non-denumerable number of rows and columns. But by saying this we make our exposition basically a-historical, for diagonalization was first used to prove the very existence of non-denumerable cardinalities.

2. Cardinality issues

A straightforward application of diagonalization shows that however we order infinite sequences of natural numbers into a succession, the succession will not contain all the sequences.

	1	2	3	4	5	...	AD
1	5	0	3	8	4	...	**1**
2	1	1	3	3	6	...	**2**
3	3	7	3	7	7	...	**1**
4	4	4	4	1	1	...	**2**
5	9	6	7	3	2	...	**1**
...	...	...	...	...	...	...	...

This is usually interpreted in such a way that there are more sequences of natural numbers than there are natural numbers; hence, that there is an infinity greater than the infinity of natural numbers.[2] Note that this result keeps holding even if we only consider sequences of some restricted subset of natural numbers, such as $\{0,1,2,3,4,5,6,7,8,9\}$ or indeed $\{0,1\}$.

Now consider real numbers between zero and one, i.e. numbers of the shape $0{,}x_1x_2x_3$ ..., where x_1, x_2, x_3 ... is an infinite sequence of one-digit numerals. Each such number can therefore be identified with an infinite sequence of natural numbers;[3] it follows that there are more real numbers than natural numbers. The acceptance of this view by Cantor [1874] marked a break in the foundations of mathematics.

2 Though this is nowadays an almost universally accepted interpretation, it is perhaps not quite inevitable – we might for example insist that the reason that we are not able to order all the sequences in a single succession is not a matter of their number, but rather of some peculiarities of the ordering procedure.

3 In fact, with some trivial exceptions: a real number of the shape $0{,}x_1x_2x_3...x_n999...$ (with no other digit than 9 thereafter) is considered to be the same as $0{,}x_1x_2x_3...x_n'000...$ (with zeros thereafter), where $x_n'=x_n+1$. But it is easy to show that these exceptions are not relevant.

Still more generally, take the labels of rows to be elements of an arbitrary set S and the columns of the table as the characteristic functions of its subsets: i.e. every column represents that subset of S which consists of those elements of S to which it assigns the value 1. If $S = \{e_1, e_2, e_3, ...\}$ and we denote its subsets as $s_1, s_2, s_3, ...$, we have the following table:

S	s_1	s_2	s_3	s_4	s_5	...	AD
e_1	0	1	1	0	1	...	**1**
e_2	0	0	1	0	0	...	**1**
e_3	0	0	1	0	1	...	**0**
e_4	0	0	0	1	0	...	**0**
e_5	0	0	0	1	1	...	**0**
...	...	...	...	...	...	...	...

Here diagonalization shows us that there are more subsets of any set than the elements of the set. These results are connected with the birth of set theory and especially, again, with the research of Cantor [1890].

3. Paradoxes

Suppose we have all the properties that there are (*being black*, *being a fish*, *being a color*, *being a property*, ...). Denote them as $p_1, p_2, p_3, ...$. Make them the labels of both rows and columns of a table and fill the cell in the intersection of the ith row with the jth column with 1 iff the ith property has the jth property (hence if, e.g., *being black is a color*, or if *being a property is a fish*) and with 0 otherwise; and construct the antidiagonal.

	p_1	p_2	p_3	p_4	p_5	...	p^*
p_1	0	1	1	0	1	...	**1**
p_2	0	0	1	0	0	...	**1**
p_3	0	0	1	0	1	...	**0**
p_4	0	0	0	1	0	...	**0**
p_5	0	0	0	1	1	...	**0**
...	...	...	...	...	...	...	...

Now the antidiagonal contains 1 in its ith row iff p_i does not have itself; hence, it corresponds to a property p^* that maps every property on 1 iff the property does not have itself. In this sense, p^* would seem to be the property of *not having itself*. And as this indeed appears to be a property and as we assumed that the table contained all the properties that there are, p^* must be identical with p_j for some j. Then, however, the jth element of the diagonal is the jth element of both the column p_j and the column p^*. In other words, it is $p_j(p_j)$ and at the same time it is $\neg p_j(p_j)$. Thus, $p_j(p_j)$ is true iff it is false. This is the celebrated paradox of Russell [1908].

What to make of it? One interpretation of this fact is that *not having itself* only *seems* to be a property, but in fact it is not. For how could it be a property if it cannot be a member of any set of properties? Another interpretation is that there is such a property and that hence there are real inconsistencies plaguing natural language[4] and that the role of logic is to establish artificial languages gerrymandered in such a way that no inconsistencies are let into them. Hence, let us now turn to the case when $p_1, p_2, p_3, \ldots$ are all properties *that are expressible in terms of a fixed language*.

4. Incompleteness of arithmetic

Let now $p_1, p_2, p_3, \ldots$ be not properties, but formulas of the language of Peano arithmetic (PA), each of which contains exactly one free variable. We will call such formulas *pseudopredicates*; they can clearly be considered as expressing (numerical) properties. In particular, every such formula is true of some numbers and false for others and expresses the property which a number has iff the formula is true of it. Thus, for example, the formula $x>5$ expresses the property of being bigger than five, whereas the formula $\exists y\ (x = 2.y)$ expresses the property of being even. If we denote this last formula as p, we shall denote by $p(1)$,$p(2)$, $p(365)$, etc., the result of the replacement of its single free variable by 1, 2, 365, etc., respectively, i.e. the respective formulas $\exists y\ (1 = 2.y)$, $\exists y\ (2 = 2.y)$, $\exists y\ (365 = 2.y)$, etc.

At the same time, if we fix a Gödel numbering,[5] every such pseudopredicate p will have a number $\ulcorner p \urcorner$. Then if, for instance, p is $\exists y\ (x = 2.y)$ and the Gödel number $\ulcorner p \urcorner$ of this formula is 365, we can form the formula $\exists y\ (365 = 2.y)$ (which is, by the way, obviously false), which results from replacing the only free variable of p by its own Gödel number; i.e. it is the formula $p(\ulcorner p \urcorner)$. Then, if we denote the truth value of a (closed) formula f as $|f|$ and the opposite value as $\overline{|f|}$, we can form the table such that the number in the intersection of the ith row and the jth column indicates whether the number $\ulcorner p_i \urcorner$ has the property p_j, i.e. it is the truth value of the formula $p_j(\ulcorner p_i \urcorner)$ (which is the formula that results from the replacement of all occurrences of the single free variable of p_j by the numeral $\ulcorner p_i \urcorner$):

4 In an extreme form this may lead to the theory of dialetheism (see Priest [1998]), according to which there really are propositions that are both true and false.

5 As Gödel showed, the expressions of the language of arithmetic can be "enumerated", i.e. mapped on numerals in such a way that the mapping is one-to-one and that we are able to compute the number of any formula and find the formula with any given number.

	p_1	p_2	p_3	…	AD
p_1	$\vert p_1(\ulcorner p_1 \urcorner)\vert$	$\vert p_2(\ulcorner p_1 \urcorner)\vert$	$\vert p_3(\ulcorner p_1 \urcorner)\vert$	…	$\overline{\vert \boldsymbol{p}_1(\ulcorner \boldsymbol{p}_1 \urcorner)\vert}$
p_2	$\vert p_1(\ulcorner p_2 \urcorner)\vert$	$\vert p_2(\ulcorner p_2 \urcorner)\vert$	$\vert p_3(\ulcorner p_2 \urcorner)\vert$	…	$\overline{\vert \boldsymbol{p}_2(\ulcorner \boldsymbol{p}_2 \urcorner)\vert}$
p_3	$\vert p_1(\ulcorner p_3 \urcorner)\vert$	$\vert p_2(\ulcorner p_3 \urcorner)\vert$	$\vert p_3(\ulcorner p_3 \urcorner)\vert$	…	$\overline{\vert \boldsymbol{p}_3(\ulcorner \boldsymbol{p}_3 \urcorner)\vert}$
p_4	$\vert p_1(\ulcorner p_4 \urcorner)\vert$	$\vert p_2(\ulcorner p_4 \urcorner)\vert$	$\vert p_3(\ulcorner p_4 \urcorner)\vert$	…	$\overline{\vert \boldsymbol{p}_4(\ulcorner \boldsymbol{p}_4 \urcorner)\vert}$
p_5	$\vert p_1(\ulcorner p_5 \urcorner)\vert$	$\vert p_2(\ulcorner p_5 \urcorner)\vert$	$\vert p_3(\ulcorner p_5 \urcorner)\vert$	…	$\overline{\vert \boldsymbol{p}_5(\ulcorner \boldsymbol{p}_5 \urcorner)\vert}$
…	…	…	…	…	**…**

The antidiagonal now maps $\ulcorner p_i \urcorner$ on the truth value opposite to that of $p_i(\ulcorner p_i \urcorner)$, and it is immediately clear that no formula of arithmetic can yield this mapping. And, unlike the case of *not having itself* considered in the context of all conceivable properties, this conclusion is not problematic – on the contrary, it is good for arithmetic to be put together so that it avoids the paradox.

Now Gödel showed, among other things, that the function mapping the number of a formula p on the number of $p(\ulcorner p \urcorner)$ can be expressed by a term with a free variable of the language of PA – we can introduce the function symbol Dg that expresses it. Suppose that we have a pseudopredicate Tr such that is true precisely of numbers of true formulas. In this case, we could form the formula $\neg Tr(Dg(x))$, which would produce precisely the antidiagonal column – indeed, it would be true of a formula p just in case $p(\ulcorner p \urcorner)$ would not be true. It follows that the language of PA cannot contain the pseudopredicate Tr. (This result is sometimes referred to as *Tarski's theorem*.)

On the other hand Gödel showed that there is a pseudopredicate which is true (and provably so) precisely of numbers of formulas *provable* within PA, and that hence we can introduce the predicate symbol Pr with this property. Hence we do have the formula $\neg Pr(Dg(x))$ which appears to be an analogue of the previous one with provability in place of truth. So consider a variation of the previous table in which the number in the cell at the intersection of the ith row and the jth column now indicates whether the number $\ulcorner p_i \urcorner$ has the property p_j *provably*, i.e. it is 1 iff the formula $p_j(\ulcorner p_i \urcorner)$ is provable (i.e. *provably* true) and is 0 iff it is *refutable* (i.e. *provably* false, its negation being provable). If $\Vert f \Vert$ is 1 for a provable f and is 0 for a refutable f (and $\overline{\Vert f \Vert}$ is the opposite value), we have

	p_1	p_2	p_3	…	AD
p_1	$\Vert p_1(\ulcorner p_1 \urcorner)\Vert$	$\Vert p_2(\ulcorner p_1 \urcorner)\Vert$	$\Vert p_3(\ulcorner p_1 \urcorner)\Vert$	…	$\overline{\Vert \boldsymbol{p}_1(\ulcorner \boldsymbol{p}_1 \urcorner)\Vert}$
p_2	$\Vert p_1(\ulcorner p_2 \urcorner)\Vert$	$\Vert p_2(\ulcorner p_2 \urcorner)\Vert$	$\Vert p_3(\ulcorner p_2 \urcorner)\Vert$	…	$\overline{\Vert \boldsymbol{p}_2(\ulcorner \boldsymbol{p}_2 \urcorner)\Vert}$
p_3	$\Vert p_1(\ulcorner p_3 \urcorner)\Vert$	$\Vert p_2(\ulcorner p_3 \urcorner)\Vert$	$\Vert p_3(\ulcorner p_3 \urcorner)\Vert$	…	$\overline{\Vert \boldsymbol{p}_3(\ulcorner \boldsymbol{p}_3 \urcorner)\Vert}$
p_4	$\Vert p_1(\ulcorner p_4 \urcorner)\Vert$	$\Vert p_2(\ulcorner p_4 \urcorner)\Vert$	$\Vert p_3(\ulcorner p_4 \urcorner)\Vert$	…	$\overline{\Vert \boldsymbol{p}_4(\ulcorner \boldsymbol{p}_4 \urcorner)\Vert}$
p_5	$\Vert p_1(\ulcorner p_5 \urcorner)\Vert$	$\Vert p_2(\ulcorner p_5 \urcorner)\Vert$	$\Vert p_3(\ulcorner p_5 \urcorner)\Vert$	…	$\overline{\Vert \boldsymbol{p}_5(\ulcorner \boldsymbol{p}_5 \urcorner)\Vert}$
…	…	…	…	…	**…**

Now does the pseudopredicate $\neg Pr(Dg(x))$ produce this antidiagonal? We know this cannot be the case – if it were, then we would have a contradiction, for the antidiagonal is different from all the columns in the table, and yet as $\neg Pr(Dg(x))$ is a pseudopredicate of the language of PA, it would have to be one of the columns. Why this is not the case?

It is the case that if $||p(\ulcorner p \urcorner)|| = 1$, i.e. if p is provably true of itself, $\neg Pr(Dg(\ulcorner p \urcorner))$ is provably false and hence $||\neg Pr(Dg(\ulcorner p \urcorner))|| = \overline{||p(\ulcorner p \urcorner)||} = 0$. (This follows from the fact that $p(\ulcorner p \urcorner)$ is provable iff $Pr(\ulcorner p(\ulcorner p \urcorner) \urcorner)$ is provable, and $Pr(\ulcorner p(\ulcorner p \urcorner) \urcorner)$ is equivalent to $Pr(Dg(\ulcorner p \urcorner))$ and hence to $\neg\neg Pr(Dg(\ulcorner p \urcorner))$). Conversely, if $||p(\ulcorner p \urcorner)|| = 0$, then $\overline{||p(\ulcorner p \urcorner)||} = 1$. Hence the new column contains 1 iff the diagonal contains 0. Thus, the new column would be the antidiagonal – and the contradiction would be inevitable – if it were the case that any formula were provably true iff it were not provably false. (For in this case all cells of the diagonal which would not contain 1's would contain 0's and the corresponding cells of the new column would contain 1's.) But while no formula is at the same time provable and refutable (at least as long as PA is consistent), it need not be the case that every formula is either provable, or refutable. And we see that it *cannot* be the case, in pain of contradiction. Hence if PA is consistent, then it is not complete, in pain of contradiction. This is the celebrated *incompleteness* discovered and proven by Gödel [1931].

5. Fixed points

Let us investigate an alternative way of reaching incompleteness via diagonalization, a way that is closer to the way Gödel himself proceeded. Consider a property q of numbers, i.e. a mapping of numbers on truth values. Let us form a column, p^*, by associating every pseudopredicate p with q applied to $Dg(\ulcorner p \urcorner)$:

	p^*
p_1	$q(\ulcorner p_1(\ulcorner p_1 \urcorner) \urcorner)$
p_2	$q(\ulcorner p_2(\ulcorner p_2 \urcorner) \urcorner)$
p_3	$q(\ulcorner p_3(\ulcorner p_3 \urcorner) \urcorner)$
p_4	$q(\ulcorner p_4(\ulcorner p_4 \urcorner) \urcorner)$
p_5	$q(\ulcorner p_5(\ulcorner p_5 \urcorner) \urcorner)$
...	...

Whether this column coincides with one of the columns of the table or not depends on the specific nature of q, in particular on whether q is expressible in the language of PA (in view of the obvious fact that $q(\ulcorner p_i(\ulcorner p_i \urcorner) \urcorner)$ is expressible in arithmetic just in case q is). If, for instance, q is *is not true*, then the column becomes the antidiagonal. On the other hand, if q is expressible in arithmetic, then there must be a p_j which expresses $q(\ulcorner p_i(\ulcorner p_i \urcorner) \urcorner)$.

In this case, consider the cell in the intersection of the jth row and the jth column. According to the definition of the table, it will contain the truth value of $p_j(\ulcorner p_j \urcorner)$. On the other hand, in view of the fact that this column coincides with that for p^*, it will also contain the truth value of $q(\ulcorner p_j(\ulcorner p_j \urcorner) \urcorner)$. As a result, the values of $q(\ulcorner p_j(\ulcorner p_j \urcorner) \urcorner)$ and $p_j(\ulcorner p_j \urcorner)$ are bound to coincide; schematically $q(\ulcorner p_j(\ulcorner p_j \urcorner) \urcorner) \leftrightarrow p_j(\ulcorner p_j \urcorner)$. Hence, we have shown what is usually called the *fix point lemma*: for every property q expressible in arithmetic there will be a sentence s_q of arithmetic such that $q(\ulcorner s_q \urcorner) \leftrightarrow s_q$.

In this way, we arrive at the inexpressibility of the truth property in arithmetic by an alternative route. If the property were expressible, then its negation would be also and it would have a fixed point $s_{\neg Tr}$ such that $\neg Tr(\ulcorner s_{\neg Tr} \urcorner) \leftrightarrow s_{\neg Tr}$ and hence that $Tr(\ulcorner s_{\neg Tr} \urcorner) \leftrightarrow \neg s_{\neg Tr}$. But as Tr is a truth predicate only if $Tr(\ulcorner s \urcorner) \leftrightarrow s$ for every statement s[6], it is also the case that $Tr(\ulcorner s_{\neg Tr} \urcorner) \leftrightarrow s_{\neg Tr}$. Putting the two equivalences together, we have $s_{\neg Tr} \leftrightarrow \neg s_{\neg Tr}$; and hence we have a contradiction.

Now imagine that we take q to be the property of non-provability, i.e. a property which a number has iff it is a number of a formula not provable in PA. We already know that this property is expressible in arithmetic, so it *does* have a fixed point. Hence, there is a $s_{\neg Pr}$ so that $\neg Pr(\ulcorner s_{\neg Pr} \urcorner) \leftrightarrow s_{\neg Pr}$. Now suppose that $s_{\neg Pr}$ is provable; if so, then so is $\neg Pr(\ulcorner s_{\neg Pr} \urcorner)$. But this could only be if $s_{\neg Pr}$ were not provable, hence the assumption of the provability of $s_{\neg Pr}$ leads to the contradictory conclusion of its non-provability; hence, $s_{\neg Pr}$ cannot be provable. Suppose, then, that $s_{\neg Pr}$ is refutable, hence that $\neg s_{\neg Pr}$ is provable. Then $Pr(\ulcorner s_{\neg Pr} \urcorner)$ is provable, and, as a result, $s_{\neg Pr}$ is provable. Hence $s_{\neg Pr}$ cannot be refutable either – in pain of inconsistency.

6. Turing machines

The problem of the decidability of an axiomatic system is the problem of whether we can always decide if a given formula of the system is a theorem. Note that if the system is such that every formula that is not provable is refutable, then the decision procedure is always at hand: we use the axioms and rules to continue generating the theorems and, sooner or later, we must reach either the formula, or its negation. (True, it might be a procedure that is not very practical since reaching the result may take a lot of time, but it works.) If, on the other hand, this is not the case (and in case of languages of pure logic it cannot be the case, for their theorems are only logical truths, and certainly not every negation of a sentence that is not logically true is logically true), the existence of a decision procedure is not guaranteed.

Alan Turing [1937], when he dealt with this problem, saw the necessity of exactly specifying what is a "procedure" or an "algorithm". His answer to this question was the abstract machines which later came to bear his name: *Turing machines*. For simplicity's sake, let us assume that the machines deal only with natural numbers, i.e. that if any such machine is fed with a natural number it starts computing and, if it halts, it yields another natural number. Thus, any such machine "realizes" a function from natural numbers to natural

6 The fact that this is precisely what characterized the property of truth was argued for by Tarski [1932].

numbers. We will not talk about the inner structure of the machines here, but we note that any such machine is uniquely describable by language and hence can be identified with a certain (sometimes perhaps very long) expression. Thus, all the machines can be enumerated (M_1, M_2, ...) and we can also always find the *n*th machine according to the enumeration.

Now consider the table with rows labeled with natural numbers and columns labeled with Turing machines. The number in the intersection of the *i*th row and the *j*th column is the value yielded by M_j for the input *i* (as the machine may not stop, the cell may be also empty).[7] Construct an antidiagonal as indicated in the table (where we take $M_i[i]+1$ to be 0 iff M_i does not stop for the input *i*):

	M_1	M_2	M_3	M_4	M_5	...	?
1	$M_1[1]$	$M_2[1]$	$M_3[1]$	$M_4[1]$	$M_5[1]$	...	$M_1[1]+1$
2	$M_1[2]$	$M_2[2]$	$M_3[2]$	$M_4[2]$	$M_5[2]$	...	$M_2[2]+1$
3	$M_1[3/]$	$M_2[3]$	$M_3[3]$	$M_4[3]$	$M_5[3]$	...	$M_3[3]+1$
4	$M_1[4]$	$M_2[4]$	$M_3[4]$	$M_4[4]$	$M_5[4]$	...	$M_4[4]+1$
5	$M_1[5]$	$M_2[5]$	$M_3[5]$	$M_4[5]$	$M_5[5]$	...	$M_5[5]+1$
...	...	...	...	...	...	...	

The antidiagonal cannot be computable by a Turing machine (since it is different from every column of the original table, which correspond to every Turing machine). But is it really not computable? Imagine the following computation: given a number *j* we find the machine M_j (we have already noted that we can do this), we let it run on the input *j*, add 1 and ... *voilà*! There is, of course, a snag. We must wait until M_j stops and yields its result; but what if it never stops? We would be waiting forever (for we would never know whether it merely has not stopped *yet*, or whether it would *never* stop). Hence, what we would need is an algorithm which would be able to tell us, for any given machine *M* and any input *i*, whether *M* ever stops for *i*.

Hence, there cannot be a Turing machine solving this *halting problem* – and insofar as we are convinced that everything that is solvable is solvable by a Turing machine, the problem is generally unsolvable. And, as it can be shown that the halting problem would be solvable if the predicate calculus were decidable (the stopping of every Turing machine

7 In fact, the result that at least some machines cannot stop for every argument can be established by means of a consideration similar to that by which we established the incompleteness of arithmetic. Imagine a *universal* Turing machine *U*, a machine that is able to simulate any Turing machine in the sense that if it gets, as its input, the description of some Turing machine *m* plus some data *d* (we will write $U\ (m \oplus d)$ (where '$\oplus$' symbolizes concatenation by means of some kind of separator) it stops just in case *m* stops for the input *d* and in that case it yields the same value: $U(m \oplus d) = m(d)$. It is easy to turn *U* into a machine U' such that $U'(m \oplus d) \neq m(d)$ whenever *m* stops for *d*. Further, it is easy to turn U' into U'' such that $U''(d) = U'(d \oplus d)$. Now $U''(U'') = U'(U'' \oplus U'') \neq U''(U'')$. This shows that U'' can never stop for the data U''.

turns out to be equivalent to a certain formula being true – see Boolos et al. [2002]), predicate calculus is undecidable.

7. Conclusion

Diagonalization is, in essence, a trivial method of constructing, for a square table, a column that is not yet contained in the table. However, it has far-reaching consequences; in fact, consequences that reach as far as the most path-breaking problems and results of modern logic. It allows us to extend the trivial observation that there are more subsets than elements of a set from the finite case to infinite ones, thus establishing the need for a hierarchy of infinities, instantiated by the infinite of natural numbers, that of real numbers, etc. Also, it allows us to see that *not having itself* is a property of properties that is strangely anomalous in that once it is expressed in a language, it makes the language inconsistent. Within the framework of the exactly delimited language of arithmetic, this yields us, first, the consequence that the concept of truth cannot be expressed by any pseudopredicate of the language; and, second, as there is a predicate expressing the concept of provability, the consequence that the language must be incomplete. Applied to the realm of Turing machines, it further yields us the result that the halting problem for these machines must be unsolvable. In this way, the *prima facie* simple observation of the possibility to diagonalize any square table leads us to a battery of very nontrivial results constituting, as it were, the central nervous system of modern logic.

Acknowledgments

Work on this paper has been supported by Research Grant No. 17-15645S of the Czech Science Foundation. I am grateful to Pavel Arazim for valuable critical comments.

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THE INCONSISTENCY PREDICATE ON DE MORGAN LATTICES

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ABSTRACT
We consider expansions of De Morgan lattices by an additional unary predicate interpreted in each De Morgan lattice by the ideal generated by all elements of the form $a \wedge -a$, and describe the finite lattice of strict universal Horn classes of such structures, thereby extending the description of the lattice of quasivarieties of De Morgan lattices due to Pynko. We also consider the same problem for expansions of De Morgan lattices by a constant interpreted as the maximal element of this ideal whenever it exists.
Keywords: De Morgan lattice, inconsistency, Belnap–Dunn logic, quasivariety, universal Horn class

1. Introduction

The present paper is a logically motivated investigation of certain expansions of De Morgan lattices. Before dealing with the technical side of the matter, let us therefore first outline the logical motivation.

The four-valued Belnap–Dunn logic [2, 4] was devised to handle inferences from inconsistent premises in a non-trivial way. It validates double negation introduction and elimination and the De Morgan laws without validating either the law of excluded middle or *ex contradictione quodlibet*. The Belnap–Dunn logic has a transparent four-valued semantics which preserves the truth and falsehood conditions of classical logic but allows propositions to be both true and false (corresponding to inconsistent information) or neither true nor false (corresponding to incomplete information). An essential feature of this logic is that consistent and inconsistent theories are treated on a par.

An algebraic semantics of the four-valued Belnap–Dunn logic is provided by De Morgan lattices, i.e. distributive lattices equipped with an order-inverting involution, called the De Morgan negation here. Each De Morgan lattice may be interpreted as an algebra of propositions where the lattice ordering corresponds to the entailment relation between propositions. Then $\Gamma \vdash \Delta$ holds in the Belnap–Dunn logic if and only if the inequality $\bigwedge \Gamma \leq \bigvee \Delta$ holds in each De Morgan lattice.

If we now broaden our notion of logic slightly to subsume inferences between sequents rather than formulas, we may say that the sequent $\Gamma \vdash \Delta$ is a consequence of the sequents $\Gamma_1 \vdash \Delta_1, \dots, \Gamma_n \vdash \Delta_n$ if and only if the quasiequation $\bigwedge \Gamma_1 \leq \bigvee \Delta_1 \;\&\; \dots \;\&\; \bigwedge \Gamma_n \leq$

https://doi.org/10.14712/24647055.2017.15

$\bigvee \Delta_n \Rightarrow \bigwedge \Gamma \leq \bigvee \Delta$ holds in each De Morgan lattice. That is, the Belnap–Dunn logic in this broader sense may be identified with the quasiequational theory of De Morgan lattices.

Our intention is now to extend the expressive power of the Belnap–Dunn logic (in this broader sense) by talking not only about which propositions follow from which, but also about which propositions are inconsistent. To this end we allow the premises and conclusions of our inferences to be not only sequents, but also statements of the form "the proposition φ inconsistent". This expansion of the Belnap–Dunn logic then allows us to express principles such as "if $\varphi \wedge \psi$ is inconsistent and $-\psi$ is inconsistent, then φ is inconsistent" or "if φ is inconsistent and $-\psi$ is inconsistent, then φ entails ψ".

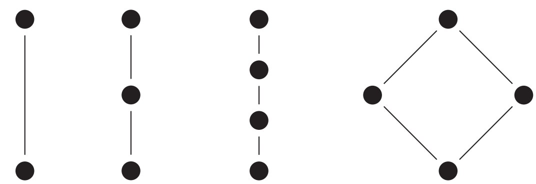
It remains to specify what we mean by calling a proposition inconsistent. There are broadly two diferent options to choose from: either we talk about *logical inconsistency* or about *material inconsistency*. In the former interpretation, a proposition is inconsistent by virtue of its logical form, e.g. all propositions of the form $(\varphi \wedge -\varphi) \vee (\psi \wedge -\psi)$ are logically inconsistent. In the latter interpretation, it is allowed that there may be propositions which are inconsistent but not merely by virtue of their logical form, i.e. inconsistency is treated as a primitive notion. These interpretations correspond to two notions of an inconsistent predicate on a De Morgan lattice: the standard inconsistency predicate, which is precisely the ideal generated by elements of the form $a \wedge -a$, and an arbitrary inconsistency predicate, which is any ideal extending the standard inconsistency predicate.

The main question which we answer here is: how many strict universal Horn classes of De Morgan lattices expanded by the standard inconsistency predicate (i.e. a unary predicate interpreted on each De Morgan lattice as the ideal generated by elements of the form $a \wedge -a$) are there?[1] For De Morgan lattices, this question was already answered by Pynko [7], who proved that there are only finitely many strict universal Horn classes (i.e. quasivarieties) of De Morgan lattices, and provided a full description of the finite lattice of quasivarieties of De Morgan lattices. By contrast, it was proved by Adams and Dziobiak [1] that there is a continuum of quasivarieties of De Morgan algebras, i.e. De Morgan lattices with a bottom and top element which are part of the algebraic signature of such lattices. In this paper, we extend Pynko's result and show that in this respect De Morgan lattices expanded by the standard inconsistency predicate behave like De Morgan lattices rather than De Morgan algebras, i.e. we prove that there are only finitely many strict universal Horn classes of such structures.

The structure of the paper is as follows. After reviewing some known facts about De Morgan lattices in Section 2, we properly define the notions sketched above and introduce several classes of De Morgan lattices with an inconsistency predicate (called De Morgan $\mathcal{I}$-lattices throughout the paper for simplicity) in Section 3. The main result of Section 4 then states that these classes in fact exhaust all of the classes which are definable by means of strict universal Horn sentences with the help of the standard inconsistency predicate. In Section 4 we also consider expansions of De Morgan lattices by an inconsistency constant (called De Morgan 0-lattices for simplicity), which is interpreted as the maximal element of the inconsistency predicate whenever it exists. It is proved that there are only

[1] Recall that a *strict universal Horn class* is a class defined by strict universal Horn sentences, i.e. disjunctions of finitely many negated atomic formulas (possibly none) and a single atomic formula in a given signature, which may in general contain relational symbols other than the equality sign.

FIGURE 1. Some De Morgan lattices



$\mathbf{B_2}$ $\mathbf{K_3}$ $\mathbf{RegK_4}$ $\mathbf{DM_4}$

finitely many strict universal Horn classes of Kleene lattices (but not of De Morgan lattices) with a standard inconsistency constant. The paper then concludes with some open questions.

2. Preliminaries

A *De Morgan lattice* is a distributive lattice equipped with an order-inverting involution, i.e. a unary operation denoted $-$ which satisfies the equations $--x = x$ and $-(x \vee y) = -x \wedge -y$, or equivalently $-(x \wedge y) = -x \vee -y$.

Figure 1 depicts some important De Morgan lattices. In all cases the De Morgan negation consists in turning the lattice upside down around the horizontal axis. In particular, note that $\mathbf{DM_4}$ is not isomorphic as a De Morgan lattice to $\mathbf{B_2} \times \mathbf{B_2}$. The algebras $\mathbf{B_2}$, $\mathbf{K_3}$, and $\mathbf{DM_4}$ are the only three subdirectly irreducible De Morgan lattices.

Definition 2.1. *A De Morgan lattice is* Boolean *if it satisfies the equation $x \wedge -x \leq y$, and it is* Kleene *if it satisfies the equation $x \wedge -x \leq y \vee -y$.*

Definition 2.2. *A De Morgan lattice is* non-idempotent *if it satisfies the quasiequation $x = -x \Rightarrow x = y$, and it is* Kleene-regular *if it satisfies the quasiequation $x \leq -x \;\&\; -x \wedge y \leq x \vee -y \Rightarrow y \leq -y$.*

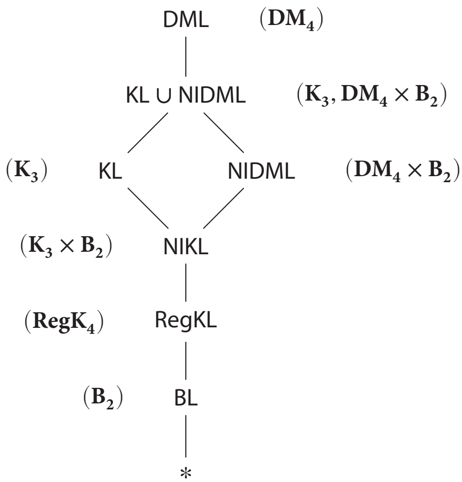
Here we follow the terminology introduced by Pynko [7], except what he calls regular Kleene lattices we shall (temporarily) call Kleene-regular lattices. We shall show later that being Kleene-regular is equivalent to being Kleene and satisfying a condition which we shall call regularity.

The following two facts will be useful later. The first one is originally due to Belnap and Spencer [3]. It was also proved as Lemma 4.3 in [7]. The second one was established in the course of the proof of Theorem 4.8 of [7].

Lemma 2.3. *A non-trivial De Morgan lattice $\mathbf{A}$ is non-idempotent if and only if there is a De Morgan lattice homomorphism $h : \mathbf{A} \to \mathbf{B_2}$.*

Lemma 2.4. *If a De Morgan lattice $\mathbf{A}$ is neither Kleene nor non-idempotent, then $\mathbf{DM_4}$ is a subalgebra of $\mathbf{A}$.*

Figure 2. The lattice of quasivarieties of De Morgan lattices



A *strict universal Horn sentence* is a disjunction of finitely many negated atomic formulas (possibly none) and exactly one atomic formula. A *strict universal Horn class* is then a class of similar structures axiomatized by a set of strict universal Horn sentences. Equivalently (see [6]), a strict universal Horn class is a similar class of structures closed under substructures and non-empty products and ultraproducts which contains the trivial structure (the singleton structure in which each atomic predicate is interpreted by a non-empty relation). We follow Gorbunov [6] in using the term *quasiequation* for strict universal Horn sentences and the terms *quasiequational class* or *quasivariety* for strict universal Horn classes, even in a signature which contains relational symbols other than the equality sign.

Note that, like Pynko [7], we shall in fact not rely on the assumption of finitarity and closure under ultraproducts. Instead of strict universal Horn sentences, we could therefore discuss their infinitary counterparts, and instead of strict universal Horn classes, we could talk about implicational classes in essentially the same sense as Pynko. The word "quasivariety" may thus be replaced by "implicational class" throughout the paper. For the sake of its familiarity, however, we stick to the language of quasivarieties.

The lattice of quasivarieties of De Morgan lattices was described by Pynko [7], who also showed that each implicational class of De Morgan lattices is a quasivariety. This lattice is shown in Figure 2 along with the generators of each quasivariety. The notation BL, RegKL, NIKL, KL, NIDML, and DML refers to the classes of Boolean lattices, Kleene-regular lattices, non-idempotent Kleene lattices, Kleene lattices, non-idempotent De Morgan lattices, and De Morgan lattices, respectively. The class KL ∪ NIDML is a quasivariety axiomatized by the quasiequation $x = -x \Rightarrow y \wedge -y \leq z \vee -z$, and the node labelled $*$ corresponds to the trivial quasivariety.

Finally, let us emphasize the distinction between De Morgan lattices and De Morgan algebras. A *De Morgan algebra* is a De Morgan lattice with a bottom and top element which are part of the algebraic signature. Each finite De Morgan lattice may thus be expanded to a De Morgan algebra. However, as noted in the introduction, this seemingly innocent addition of the bottom and top elements to the signature has a dramatic impact on the lattice of quasivarieties: there is a continuum of quasivarieties of De Morgan (in fact, Kleene) algebras but there are only finitely many quasivarieties of De Morgan lattices. We emphasize that throughout the present paper we deal with De Morgan *lattices*. The only results which in any way involve De Morgan algebras and their expansions are Propositions 4.6 and 4.9.

3. The inconsistency predicate

As we already observed in the introduction, each De Morgan lattice $\mathbf{A}$ comes equipped with an ideal, denoted $\mathcal{I}_{\mathbf{A}}$, generated by the elements of the form $a \wedge -a$ for $a \in \mathbf{A}$, or equivalently by the elements $a \in \mathbf{A}$ such that $a \leq -a$. If $\mathbf{A}$ is Kleene, then the ideal $\mathcal{I}_{\mathbf{A}}$ admits a particularly simple description.

Lemma 3.1. *If* $\mathbf{A}$ *is a Kleene lattice, then* $\mathcal{I}_{\mathbf{A}} = \{a \in \mathbf{A} \mid a \leq -a\}$.

Proof. Clearly $\{a \in \mathbf{A} \mid a \leq -a\} \subseteq \mathcal{I}_{\mathbf{A}}$. Conversely, suppose that $a = (a_1 \wedge -a_1) \vee \ldots \vee (a_n \wedge -a_n) \nleq (a_1 \vee -a_1) \wedge \ldots \wedge (a_n \vee -a_n) = -a$ for some $a \in \mathbf{A}$. Then there are a_i and a_j such that $a_i \wedge -a_i \nleq a_j \vee -a_j$, hence $\mathbf{A}$ is not Kleene. □

If $\mathbf{A}$ is a De Morgan lattice and $\mathcal{I}$ is an ideal on $\mathbf{A}$ such that $\mathcal{I}_{\mathbf{A}} \subseteq \mathcal{I}$, then the structure $(\mathbf{A}, \mathcal{I})$ will be called a *De Morgan lattice with an inconsistency predicate* or a *De Morgan* $\mathcal{I}$*-lattice* for short. De Morgan $\mathcal{I}$-lattices form a quasivariety which is axiomatized relative to the quasivariety of De Morgan lattices by the quasiequations $a \in \mathcal{I} \;\&\; b \in \mathcal{I} \Rightarrow a \vee b \in \mathcal{I}$ and $a \wedge -a \in \mathcal{I}$.

A De Morgan $\mathcal{I}$-lattice $(\mathbf{A}, \mathcal{I})$ will be called *standard* if $\mathcal{I} = \mathcal{I}_{\mathbf{A}}$, and it will be called *totally inconsistent* if the ideal $\mathcal{I}$ is the whole of $\mathbf{A}$. Each De Morgan lattice therefore has a unique standard expansion and a unique totally inconsistent expansion.

We shall now define several quasivarieties of De Morgan $\mathcal{I}$-lattices and show that they correspond, in a natural sense, to the quasivarieties of De Morgan lattices defined in the previous section.

Definition 3.2. *A De Morgan* $\mathcal{I}$*-lattice is:*

- Boolean *if it satisfies* $x \in \mathcal{I} \Rightarrow x \leq y$
- Kleene *if it satisfies* $x \in \mathcal{I} \;\&\; -y \in \mathcal{I} \Rightarrow x \leq y$
- non-idempotent *if it satisfies* $x \in \mathcal{I} \;\&\; -x \in \mathcal{I} \Rightarrow x = y$
- regular *if it satisfies* $x \wedge y \in \mathcal{I} \;\&\; -y \in \mathcal{I} \Rightarrow x \in \mathcal{I}$

Proposition 3.3. *Each Kleene* $\mathcal{I}$*-lattice is standard.*

Proof. Suppose that $a \in \mathcal{I}$ holds in a Kleene $\mathcal{I}$-lattice $(\mathbf{A}, \mathcal{I})$. Then clearly $-(-a) \in \mathcal{I}$, therefore by the definition of a Kleene $\mathcal{I}$-lattice we have $a \leq -a$, i.e. $a = a \wedge -a \in \mathcal{I}_{\mathbf{A}}$. □

Proposition 3.4. *A non-trivial De Morgan $\mathcal{I}$-lattice is totally inconsistent if and only if it is regular but not non-idempotent.*

Proof. Each totally inconsistent De Morgan $\mathcal{I}$-lattice $(\mathbf{A}, \mathcal{I})$ is clearly regular, and if it is non-idempotent, then it must be trivial, since $a \in \mathcal{I}$ and $-a \in \mathcal{I}$ for all $a \in \mathbf{A}$. Conversely, if $(\mathbf{A}, \mathcal{I})$ is not non-idempotent, then is some $a \in \mathbf{A}$ such that both $a \in \mathcal{I}$ and $-a \in \mathcal{I}$. If $(\mathbf{A}, \mathcal{I})$ is moreover regular, then taking $y = a$ in the regularity quasiequation yields that the De Morgan $\mathcal{I}$-lattice $(\mathbf{A}, \mathcal{I})$ is totally inconsistent. □

The correspondence between De Morgan lattices and their standard expansions is immediate for Boolean, Kleene, and non-idempotent lattices.

Proposition 3.5. *A De Morgan lattice is Boolean (Kleene, non-idempotent) if and only if its standard expansion is Boolean (Kleene, non-idempotent).*

Proof. If $\mathbf{A}$ is Boolean, then $\mathcal{I}_{\mathbf{A}} = \{\bot\}$, hence $a \in \mathcal{I}_{\mathbf{A}}$ implies $a \leq b$. Conversely, if $\mathcal{I}_{\mathbf{A}} = \{\bot\}$, then $a \wedge -a \leq b$ since $a \wedge -a \in \mathcal{I}_{\mathbf{A}}$ for all $a, b \in \mathbf{A}$.

If $\mathbf{A}$ is Kleene, then by Lemma 3.1 the condition $a \in \mathcal{I}_{\mathbf{A}}$ and $-b \in \mathcal{I}_{\mathbf{A}}$ implies that $a \leq -a$ and $-b \leq b$, hence $a = a \wedge -a \leq b \vee -b = b$. Conversely, if $a \in \mathcal{I}_{\mathbf{A}}$ and $-b \in \mathcal{I}_{\mathbf{A}}$ together imply that $a \leq b$, then in particular $a \wedge -a \leq b \vee -b$ holds for all $a, b \in \mathbf{A}$.

If the standard expansion of $\mathbf{A}$ is non-idempotent and $\mathbf{A}$ is non-trivial, then clearly the De Morgan negation on $\mathbf{A}$ cannot have any fixpoint. Conversely, suppose that $\mathbf{A}$ is a non-trivial non-idempotent De Morgan lattice. Then there is a homomorphism $h : \mathbf{A} \to \mathbf{B_2}$ by Lemma 2.3. But $\mathcal{I} \subseteq h^{-1}\{\bot\}$ and $-\mathcal{I} \subseteq h^{-1}\{\top\}$, hence $\mathcal{I} \cap -\mathcal{I} = \emptyset$. □

Finally, the reader will notice that the quasivariety of regular De Morgan $\mathcal{I}$-lattices defined above itself does not correspond to any of the quasivarieties of De Morgan lattices listed in the previous section. Nonetheless, we may still define a corresponding class of regular De Morgan lattices and show that its intersection with the variety of Kleene lattices is precisely the quasivariety of regular Kleene lattices introduced by Pynko [7].

Definition 3.6. *A De Morgan lattice* $\mathbf{A}$ *is* regular *if for each $a \in \mathbf{A}$ such that $a \notin \mathcal{I}_{\mathbf{A}}$ there is a homomorphism $h : \mathbf{A} \to \mathbf{B_2}$ such that $h(a) = \top$.*

It is easy to see that this definition may equivalently be stated as follows: a De Morgan algebra $\mathbf{A}$ is regular if and only if $\mathcal{I}_{\mathbf{A}} = \bigcap_{h \in \mathrm{Hom}(\mathbf{A}, \mathbf{B_2})} h^{-1}\{\bot\}$, where $\mathrm{Hom}(\mathbf{A}, \mathbf{B_2})$ denotes the set of all homomorphisms $h : \mathbf{A} \to \mathbf{B_2}$.

It will be useful to provide a description of the smallest congruence θ on a given De Morgan lattice $\mathbf{A}$ such that $\mathbf{A}/\theta$ is a Boolean lattice. We shall call such congruences *Boolean*. Each De Morgan lattice has a smallest Boolean congruence, namely the De Morgan lattice congruence generated by identifying all elements of $\mathcal{I}_{\mathbf{A}}$.

Lemma 3.7. *The smallest Boolean congruence on a De Morgan lattice* $\mathbf{A}$ *relates x and y if and only if $-a \wedge (x \vee b) = -a \wedge (y \vee b)$ for some $a, b \in \mathcal{I}_{\mathbf{A}}$.*

Proof. Let us denote this relation θ. We first verify that it is a De Morgan lattice congruence. It is clear that θ is reflexive and symmetric. Moreover, if $-a \wedge (x \vee b) = -a \wedge (y \vee b)$ and $-c \wedge (y \vee d) = -c \wedge (z \vee d)$ for $a, b, c, d \in \mathcal{I}_{\mathbf{A}}$, then $-(a \vee c) \wedge (x \vee b \vee d) = -(a \vee c) \wedge (y \vee b \vee d) = -(a \vee c) \wedge (z \vee b \vee d)$ and clearly $a \vee c \in \mathcal{I}_{\mathbf{A}}$ and $b \vee d \in \mathcal{I}_{\mathbf{A}}$.

The relation θ is therefore an equivalence relation. It is easy to see that it also respects meets. To prove that it respects De Morgan negation (and therefore also joins), suppose that $-a \wedge (x \vee b) = -a \wedge (y \vee b)$ for $a, b \in \mathcal{I}_{\mathbf{A}}$. Then $(-x \wedge -b) \vee a = (-y \wedge -b) \vee a$, hence $-(b \wedge -a) \wedge (-x \vee a) = -(b \wedge -a) \wedge (-y \vee a)$ and $-x\theta -y$. The relation θ is therefore a De Morgan lattice congruence.

If $x, y \in \mathcal{I}_{\mathbf{A}}$, then $x \vee y \in \mathcal{I}_{\mathbf{A}}$ and taking $b = x \vee y$ and arbitrary $a \in \mathcal{I}_{\mathbf{A}}$ yields $x\theta y$, therefore θ is a Boolean congruence. Conversely, each Boolean congruence identifies x and $-a \wedge (x \vee b)$ as well as y and $-a \wedge (y \vee b)$, therefore each Boolean congruence extends θ. □

Proposition 3.8. *A De Morgan lattice is regular if and only if its standard expansion is regular.*

Proof. Suppose that the implication $x \wedge y \in \mathcal{I}_{\mathbf{A}}$ & $-y \in \mathcal{I}_{\mathbf{A}} \Rightarrow x \in \mathcal{I}_{\mathbf{A}}$ holds and let θ be the smallest Boolean congruence on $\mathbf{A}$. If $x/\theta = \bot/\theta$, where $\bot/\theta$ is the bottom element of $\mathbf{A}/\theta$, then by the previous lemma there are $a, b \in \mathcal{I}_{\mathbf{A}}$ such that $-b \wedge (x \vee a) \in \mathcal{I}_{\mathbf{A}}$, hence $-b \wedge x \in \mathcal{I}_{\mathbf{A}}$. But then $x \in \mathcal{I}_{\mathbf{A}}$. This shows that $\mathcal{I}_{\mathbf{A}} = \bigcap_{h \in \mathrm{Hom}(\mathbf{A}, \mathbf{B_2})} h^{-1}\{\bot\}$.

Conversely, if a De Morgan lattice $\mathbf{A}$ is regular, then $x \in \mathcal{I}_{\mathbf{A}}$ if and only if $h(x) = \bot$ for all homomorphisms $h : \mathbf{A} \to \mathbf{B_2}$. But if $h(x \wedge y) = \bot$ and $h(-y) = \bot$, then $h(y) = \top$ and $h(x) = h(x \wedge y) = \bot$. □

Proposition 3.9. *A De Morgan lattice is both regular and Kleene if and only if it is Kleene-regular, i.e. if and only if it satisfies the quasiequation $x \leq -x$ & $-x \wedge y \leq x \vee -y \Rightarrow y \leq -y$.*

Proof. Let $\mathbf{A}$ be a De Morgan lattice which satisfies the above quasiequation. To prove that $\mathbf{A}$ is Kleene, substitute $x = u \wedge -u$ and $y = (u \wedge -u) \vee (v \wedge -v)$. Then $x = u \wedge -u \leq u \vee -u = -x$ and $-x \wedge y = (u \vee -u) \wedge ((u \wedge -u) \vee (v \wedge -v)) = (u \wedge -u) \vee ((u \vee -u) \wedge (v \wedge -v))$, while $x \vee -y = (u \wedge -u) \vee ((u \vee -u) \wedge (v \vee -v)) = (u \vee -u) \wedge ((u \wedge -u) \vee (v \vee -v))$. Therefore $-x \wedge y \leq x \vee -y$ and $y \leq -y$, i.e. $(u \wedge -u) \vee (v \wedge -v) \leq (u \vee -u) \wedge (v \vee -v)$. It follows that $u \wedge -u \leq v \vee -v$.

To prove that $\mathbf{A}$ is regular, suppose that $a/\theta = \bot/\theta$ for each Boolean congruence θ on $\mathbf{A}$, where $\bot/\theta$ is the bottom element of $\mathbf{A}/\theta$. We are to show that $a \in \mathcal{I}_{\mathbf{A}}$. Lemma 3.7 then implies that there are $b, c \in \mathcal{I}_{\mathbf{A}}$ such that $-b \wedge (a \vee c) \in \mathcal{I}_{\mathbf{A}}$, hence $-b \wedge a = d \in \mathcal{I}_{\mathbf{A}}$. But then $b \vee d \in \mathcal{I}_{\mathbf{A}}$ and $-(b \vee d) \wedge a \leq d$, hence $-(b \vee d) \wedge a \leq -a \vee b \vee d$. Taking into account that $a \in \mathcal{I}_{\mathbf{A}}$ is equivalent to $a \leq -a$ for all $a \in \mathbf{A}$ by Lemma 3.1, substituting $x = b \vee d$ and $y = a$ into the quasiequation now yields that $a \in \mathcal{I}_{\mathbf{A}}$.

Conversely, let $\mathbf{A}$ be a regular Kleene lattice with $a \leq -a$ and $b \nleq -b$, i.e. $a \in \mathcal{I}_{\mathbf{A}}$ and $b \notin \mathcal{I}_{\mathbf{A}}$ by Lemma 3.1. By regularity, there is some $h : \mathbf{A} \to \mathbf{B_2}$ such that $h(b) = \top$. But $h(a) = \bot$ and $h(-a \wedge b) = -h(a) \wedge h(b) = \top \nleq \bot = h(a) \vee -h(b) = h(a \vee -b)$, hence $-a \wedge b \nleq a \vee -b$. □

The above proposition may be thought of as an explanation of the rather non-transparent quasiequation $x \leq -x$ & $-x \wedge y \leq x \vee -y \Rightarrow y \leq -y$.

We now compile the correspondences proved above into a single theorem.

Theorem 3.10. *If* $(\mathbf{A}, \mathcal{I})$ *is Boolean (Kleene, non-idempotent), then so is the De Morgan lattice* $\mathbf{A}$*. Conversely, if* $\mathbf{A}$ *is Boolean (Kleene, non-idempotent, regular), then so is its standard expansion* $(\mathbf{A}, \mathcal{I}_{\mathbf{A}})$*.*

Proof. We have already proved that a De Morgan lattice is Boolean (Kleene, non-idempotent, regular) if and only if it standard expansion is. The first claim of the theorem now follows by virtue of the fact that $\mathcal{I}_{\mathbf{A}} \subseteq \mathcal{I}$ holds in every De Morgan $\mathcal{I}$-lattice $(\mathbf{A}, \mathcal{I})$ and the predicate $\mathcal{I}$ does not occur in the consequent of any of the quasiequations defining the classes of Boolean, Kleene, and non-idempotent De Morgan $\mathcal{I}$-lattices. □

Observe that the first part of the above theorem does not hold when it comes to regularity, a simple counter-example being the totally inconsistent expansion of any non-regular De Morgan algebra.

4. Quasivarieties of De Morgan $\mathcal{I}$-lattices

We shall now investigate precisely how much expressive power the standard inconsistency predicate adds to De Morgan lattices. The goal of the present section will be to describe the lattice of quasivarieties of *standard* De Morgan $\mathcal{I}$-lattices, and in particular to prove that it is finite.

Before we can proceed, the notion of a quasivariety of standard De Morgan $\mathcal{I}$-lattices requires some clarification. The class of all standard De Morgan $\mathcal{I}$-lattices is not closed under substructures (consider the two-element subalgebra of the standard expansion of $\mathbf{DM_4}$), in particular it is not a quasivariety. By a quasivariety of standard De Morgan $\mathcal{I}$-lattices we shall therefore *not* mean a quasivariety all of whose elements are standard De Morgan $\mathcal{I}$-lattices. Rather, we shall use the following definition.

Definition 4.1. *A class of standard De Morgan* $\mathcal{I}$*-lattices is a* quasivariety of standard De Morgan $\mathcal{I}$-lattices *if it is the intersection of a quasivariety of De Morgan* $\mathcal{I}$*-lattices and the class of all standard De Morgan* $\mathcal{I}$*-lattices.*

Quasivarieties of standard De Morgan $\mathcal{I}$-lattices may be put into one-to-one correspondence with certain quasivarieties of De Morgan $\mathcal{I}$-lattices, which are more convenient to handle using the theory of quasivarieties.

Definition 4.2. *A quasivariety of De Morgan* $\mathcal{I}$*-lattices is* standard *if it is generated by its standard elements.*

The quasivarieties of standard De Morgan $\mathcal{I}$-lattices ordered by inclusion form a lattice, as do the standard quasivarieties of De Morgan $\mathcal{I}$-lattices. In the following proposition, DMIL_{st} shall denote the class of all standard De Morgan $\mathcal{I}$-lattices.

Proposition 4.3. *The lattice of standard quasivarieties of De Morgan* $\mathcal{I}$*-lattices is isomorphic to the lattice of quasivarieties of standard De Morgan* $\mathcal{I}$*-lattices via the mapping* $\mathsf{K} \mapsto \mathsf{K} \cap \mathsf{DMIL}_{st}$.

Proof. This mapping is clearly monotonic. Moreover, if K is a quasivariety of De Morgan $\mathcal{I}$-lattices, then $\mathsf{K} \cap \mathsf{DMIL}_{st}$ is by definition a quasivariety of standard De Morgan

Table 1. The standard quasivarieties of De Morgan $\mathcal{I}$-lattices

Name	Abbreviation	Generated by
Boolean	BL	$\mathbf{B}_2^{\mathbf{s}}$
Kleene	KL	$\mathbf{K}_3^{\mathbf{s}}$
De Morgan	DML	$\mathbf{K}_3^{\mathbf{s}}$, $\mathbf{DM}_4^{\mathbf{s}}$
regular	RegDML	$\mathbf{RegK}_4^{\mathbf{s}}$, $\mathbf{DM}_4^{\mathbf{s}}$
regular Kleene	RegKL	$\mathbf{RegK}_4^{\mathbf{s}}$
non-idempotent	NIDML	$\mathbf{K}_3^{\mathbf{s}} \times \mathbf{B}_2^{\mathbf{s}}$, $\mathbf{DM}_4^{\mathbf{s}} \times \mathbf{B}_2^{\mathbf{s}}$
non-idempotent Kleene	NIKL	$\mathbf{K}_3^{\mathbf{s}} \times \mathbf{B}_2^{\mathbf{s}}$
regular non-idempotent	RegNIDML	$\mathbf{DM}_4^{\mathbf{s}} \times \mathbf{B}_2^{\mathbf{s}}$
totally inconsistent	TIL	$\mathbf{DM}_4^{\mathbf{s}}$

$\mathcal{I}$-lattices. It remains to prove that if K and L are standard quasivarieties of De Morgan $\mathcal{I}$-lattices and $\mathsf{K} \cap \mathsf{DMIL}_{st} \subseteq \mathsf{L} \cap \mathsf{DMIL}_{st}$, then $\mathsf{K} \subseteq \mathsf{L}$. Since K and L are standard, $\mathsf{K} \cap \mathsf{DMIL}_{st}$ ($\mathsf{L} \cap \mathsf{DMIL}_{st}$) generates K (L) as a quasivariety, therefore $\mathsf{K} \cap \mathsf{DMIL}_{st} \subseteq \mathsf{L} \cap \mathsf{DMIL}_{st}$ implies $\mathsf{K} \subseteq \mathsf{L}$. □

Instead of studying the lattice of quasivarieties of De Morgan lattices with a standard inconsistency predicate, we can therefore investigate the isomorphic lattice of standard quasivarieties of De Morgan lattices. In other words, although we are ultimately interested in studying the standard inconsistency predicate on each De Morgan lattice, admitting non-standard inconsistency predicates will be a useful tool in the study of De Morgan lattices equipped with the standard inconsistency predicate.

As an example illustrating the notion of a standard quasivariety, consider the quasivariety of De Morgan $\mathcal{I}$-lattices axiomatized by $x \wedge -x \leq y$. This class is not a standard quasivariety of De Morgan $\mathcal{I}$-lattices because each standard De Morgan $\mathcal{I}$-lattice which satisfies this equation also satisfies the quasiequation $x \in \mathcal{I} \Rightarrow x \leq y$, which however is not a consequence of $x \wedge -x \leq y$, as witnessed by the totally inconsistent expansion of the Boolean lattice $\mathbf{B_2}$. On the other hand, the quasivariety axiomatized by $x \in \mathcal{I} \Rightarrow x \leq y$ is standard, as it is generated by the standard expansion of $\mathbf{B_2}$.

When talking about De Morgan $\mathcal{I}$-lattices, the notation $\mathbf{A}^{\mathbf{s}}$ will be used to denote the expansion of the De Morgan lattice $\mathbf{A}$ by the standard inconsistency predicate, as in Table 1. To avoid unnecessary proliferation of indices, we introduce the harmless convention of using e.g. NIDML to denote either the quasivariety of non-idempotent De Morgan lattices or the quasivariety of non-idempotent De Morgan $\mathcal{I}$-lattices or, later on, the quasivariety of non-idempotent De Morgan 0-lattices. It will always be clear from the context which of these is the intended interpretation of e.g. NIDML.

Note that in this section we shall occasionally use the term "algebra" to refer to De Morgan $\mathcal{I}$-lattices, even though strictly speaking they are not algebras in the sense of universal algebra.

Theorem 4.4. *The quasivarieties of De Morgan $\mathcal{I}$-lattices introduced in the previous section are generated by the standard algebras shown in Table 1.*

Proof. It is easy to verify that the algebras listed in Table 1 belong to the appropriate quasivarieties. Conversely, we prove that each finite algebra which belongs to a quasivariety listed in Table 1 embeds into a product of the appropriate algebras. In particular, observe that if a finite De Morgan $\mathcal{I}$-lattice $\mathbf{A}$ is a subdirect product of the algebras $\mathbf{B}_i$ for $i \in I$, then the ideal $\mathcal{I}_{\mathbf{A}}$ is a product of the ideals $\mathcal{I}_{\mathbf{B}_i}$ for $i \in I$.

Let $\mathbf{A}$ be a non-trivial finite De Morgan $\mathcal{I}$-lattice. As a De Morgan lattice, $\mathbf{A}$ is a subdirect product of $\mathbf{B_2}$, $\mathbf{K_3}$, and $\mathbf{DM_4}$. Moreover, the non-standard (totally inconsistent) expansions of $\mathbf{B_2}$ and $\mathbf{K_3}$ are subalgebras of the standard expansion of $\mathbf{DM_4}$, therefore $\mathbf{A}$ embeds into a product of the standard expansions of $\mathbf{B_2}$, $\mathbf{K_3}$, and $\mathbf{DM_4}$.

If $\mathbf{A}$ is Kleene, then $\mathbf{A}$ is standard, hence each factor in the subdirect decomposition of $\mathbf{A}$ is standard. It now suffices to recall that each Kleene (Boolean, regular Kleene, non-idempotent Kleene) lattice embeds into a product of the appropriate De Morgan lattices.

If $\mathbf{A}$ is totally inconsistent, the claim follows immediately from the fact that each De Morgan lattice embeds into some power of $\mathbf{DM_4}$.

If $\mathbf{A}$ is non-idempotent, then the subdirect decomposition of $\mathbf{A}$ contains $\mathbf{B_2}$. If all occurrences of $\mathbf{B_2}$ in the subdirect decomposition of $\mathbf{A}$ were totally inconsistent, then $\mathbf{A}$ could not be non-idempotent as a De Morgan $\mathcal{I}$-lattice. Therefore $\mathbf{A}$ contains an occurrence of $\mathbf{B_2^s}$ in its subdirect decomposition. The claim now follows by virtue of the inclusions $\mathbf{K_3^{ti}} \times \mathbf{B_2^s} \subseteq \mathbf{DM_4^s} \times \mathbf{B_2^s}$ and $\mathbf{B_2^{ti}} \times \mathbf{B_2^s} \subseteq \mathbf{DM_4^s} \times \mathbf{B_2^s}$ and $\mathbf{B_2^s} \subseteq \mathbf{K_3^s} \times \mathbf{B_2^s}$, where $\mathbf{B_2^{ti}}$ and $\mathbf{K_3^{ti}}$ are the totally inconsistent expansions of the De Morgan lattices $\mathbf{B_2}$ and $\mathbf{K_3}$.

Finally, let $\mathbf{A}$ be regular. If $\mathbf{B_2^s}$ does not occur in the subdirect decomposition of $\mathbf{A}$, then $\mathbf{A}$ is not non-idempotent, hence it is totally inconsistent and embeds into some power of $\mathbf{DM_4^s}$. We may divide the subdirect factors of $\mathbf{A}$ into two groups (each of them possibly empty) and view $\mathbf{A}$ as a subdirect product of a totally inconsistent algebra $\mathbf{B}$ and a Kleene algebra $\mathbf{C}$. Observe now that $\mathbf{A}$ is regular only if $\mathbf{C}$ is: if a witness to the failure of regularity in $\mathbf{C}$, then $\langle a, b \rangle$ is a witness to the failure of regularity in $\mathbf{A}$ for any $\langle a, b \rangle \in \mathbf{A}$. But we have already seen that the quasivariety of regular Kleene $\mathcal{I}$-lattices is generated by $\mathbf{RegK_4^s}$.

Moreover, if $\mathbf{A}$ is non-idempotent, then as we observed above, it may contain $\mathbf{B_2^s}$ in its subdirect decomposition. Since $\mathbf{B_2^s} \subseteq \mathbf{RegK_4^s}$, we may take the generators to be $\mathbf{RegK_4^s}$ and $\mathbf{DM_4^s} \times \mathbf{B_2^s}$. ☐

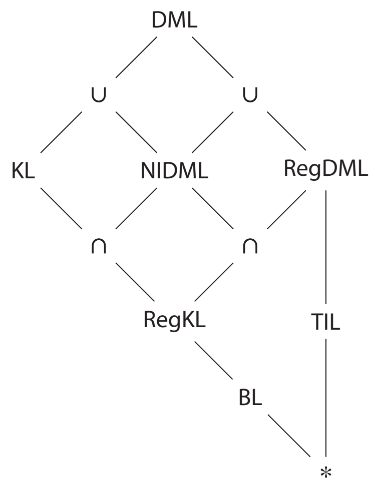
In particular, all of these quasivarieties are standard. Figure 3 depicts the lattice of these quasivarieties ordered by inclusion. The nodes which are, for the sake of easier readability, labelled ∪ or ∩ are just the unions of the quasivarieties below and the intersections of the quasivarieties above, and the node labelled $*$ is the trivial quasivariety (containing only the singleton De Morgan $\mathcal{I}$-lattice). In particular, $\mathsf{KL} \cup \mathsf{NIDML}$ is the quasivariety axiomatized by $x \in \mathcal{I}\ \&\ -x \in \mathcal{I}\ \&\ y \in \mathcal{I}\ \&\ -z \in \mathcal{I} \Rightarrow y \leq z$.

We now show that Figure 3 in fact shows *all* standard quasivarieties of De Morgan $\mathcal{I}$-lattices.

Theorem 4.5. *The lattice of standard quasivarieties of De Morgan $\mathcal{I}$-lattices is the finite lattice shown in Figure 3.*

Proof. The generating algebras listed in Table 1 witness that all of these quasivarieties are distinct and standard. It suffices to prove the following facts for each non-trivial standard quasivariety K of De Morgan $\mathcal{I}$-lattices:

FIGURE 3. The standard quasivarieties of De Morgan $\mathcal{J}$-lattices



(i) either $\mathsf{BL} \subseteq \mathsf{K}$ or $\mathsf{TIL} \subseteq \mathsf{K}$
(ii) either $\mathsf{BL} \subseteq \mathsf{K}$ or $\mathsf{K} \subseteq \mathsf{TIL}$
(iii) either $\mathsf{K} \subseteq \mathsf{BL}$ or $\mathsf{K} \subseteq \mathsf{TIL}$ or $\mathsf{RegKL} \subseteq \mathsf{K}$
(iv) either $\mathsf{K} \subseteq \mathsf{NIDML} \cup \mathsf{RegDML}$ or $\mathsf{KL} \subseteq \mathsf{K}$
(v) either $\mathsf{K} \subseteq \mathsf{RegDML}$ or $\mathsf{NIKL} \subseteq \mathsf{K}$
(vi) either $\mathsf{K} \subseteq \mathsf{KL}$ or $\mathsf{K} \subseteq \mathsf{TIL}$ or $\mathsf{RegNIDML} \subseteq \mathsf{K}$
(vii) either $\mathsf{K} \subseteq \mathsf{NIDML} \cup \mathsf{KL}$ or $\mathsf{TIL} \subseteq \mathsf{K}$

To prove (i), let $\mathbf{A}$ be a non-trivial standard De Morgan $\mathcal{J}$-lattice. If $\mathbf{A}$ is not totally inconsistent, then $\mathbf{B}_2^{\mathbf{s}} \subseteq \mathbf{A}$. If $\mathbf{A}$ is totally inconsistent, then it is clearly neither Kleene nor non-idempotent, hence $\mathbf{DM}_4^{\mathbf{s}} \subseteq \mathbf{A}$ by Lemma 2.4. Therefore either $\mathsf{BL} \subseteq \mathsf{K}$ or $\mathsf{TIL} \subseteq \mathsf{K}$.

To prove (ii), suppose that $\mathsf{K} \nsubseteq \mathsf{TIL}$. Then some $\mathbf{A} \in \mathsf{K}$ is not totally inconsistent. But then $a \notin \mathcal{J}_{\mathbf{A}}$ for some $a \in \mathbf{A}$, hence $\{a \wedge -a, a \vee -a\}$ is a subalgebra of $\mathbf{A}$ isomorphic to $\mathbf{B}_2^{\mathbf{s}}$. Therefore $\mathsf{BL} \subseteq \mathsf{K}$.

To prove (iii), suppose that $\mathsf{K} \nsubseteq \mathsf{TIL}$ and $\mathsf{K} \nsubseteq \mathsf{BL}$. Then some $\mathbf{A} \in \mathsf{K}$ is not totally inconsistent and some $\mathbf{B} \in \mathsf{K}$ is not Boolean. Let $a \in \mathbf{A}$ be such that $-a \leq a$ and $a \notin \mathcal{J}_{\mathbf{A}}$ (hence $-a < a$) and let $b, c \in \mathbf{B}$ be such that $b \leq -b$, $b \in \mathcal{J}_{\mathbf{B}}$, and $c < b$. Then $\{(-a, c), (-a, b), (a, -b), (a, -c)\}$ is a subalgebra of $\mathbf{A} \times \mathbf{B}$ isomorphic to $\mathbf{RegK}_4^{\mathbf{s}}$. Therefore $\mathsf{RegKL} \subseteq \mathsf{K}$.

To prove (iv), suppose that $\mathsf{K} \nsubseteq \mathsf{NIDML} \cup \mathsf{RegDML}$. Then some standard $\mathbf{A} \in \mathsf{K}$ is neither non-idempotent nor totally inconsistent. There is therefore some $a \in \mathbf{A}$ such that $a = -a$ and some $b \in \mathbf{A}$ such that $b \notin \mathcal{J}_{\mathbf{A}}$. Then $\{a \wedge -b, a, a \vee b\}$ is a subalgebra of $\mathbf{A}$ isomorphic to $\mathbf{K}_3^{\mathbf{s}}$. Therefore $\mathsf{KL} \subseteq \mathsf{K}$.

To prove (v), suppose that $\mathsf{K} \nsubseteq \mathsf{RegDML}$. Then $\mathbf{B}_2^{\mathbf{s}} \in \mathsf{K}$ and some standard $\mathbf{A} \in \mathsf{K}$ is not regular, i.e. there are some $a \in \mathbf{A}$ and $b \in \mathcal{J}_{\mathbf{A}}$ such that $-b \wedge a \in \mathcal{J}_{\mathbf{A}}$ but $a \notin \mathcal{J}_{\mathbf{A}}$, in particular $a \nleq -b$. Without loss of generality $b = -b \wedge a$, since $-b \wedge a \in \mathcal{J}_{\mathbf{A}}$ and

$-(-b \wedge a) \wedge a = (b \vee -a) \wedge a = b \vee (a \wedge -a) \in \mathcal{I}_{\mathbf{A}}$. We may moreover take $a = a \vee b$, since $-b \wedge (a \vee b) = (-b \wedge a) \vee (b \wedge -b) \in \mathcal{I}_{\mathbf{A}}$ and $-b \wedge (a \vee b) = (-b \wedge a) \vee (b \wedge -b) = b \vee (b \wedge -b) = b$. The De Morgan lattice $\{a \wedge -a, b, a, -a, -b, a \vee -a\}$ is a subalgebra of the De Morgan lattice reduct of $\mathbf{A}$, let us call it $\mathbf{B}$. We know that $a \notin \mathcal{I}_{\mathbf{A}}$ and clearly $a \wedge -a, b \in \mathcal{I}_{\mathbf{A}}$. If $-a \nleq a$, then $\mathbf{B}$ is isomorphic as a De Morgan lattice to $\mathbf{K_3} \times \mathbf{B_2}$, hence the algebra $\mathbf{B} \times \mathbf{B_2^s}$ has a subalgebra isomorphic to $\mathbf{K_3^s} \times \mathbf{B_2^s}$, namely $\{a \wedge -a, b, a\} \times \{\bot\} \cup \{-a, -b, a \vee -a\} \times \{\top\}$. Likewise, if $-a \leq a$, then $-a \leq -b \wedge a = b$, thus $-b \leq a$ and $b = -b \wedge a = -b$. The algebra $\mathbf{B}$ is therefore isomorphic to $\mathbf{K_3^s}$ and $\mathbf{K_3^s} \times \mathbf{B_2^s} \in \mathsf{K}$, hence $\mathsf{K} \subseteq \mathsf{NIKL}$.

To prove (vi), suppose that $\mathsf{K} \nsubseteq \mathsf{KL}$ and $\mathsf{K} \nsubseteq \mathsf{TIL}$. Then $\mathbf{B_2^s} \in \mathsf{K}$ there is some standard $\mathbf{A} \in \mathsf{K}$ which is not Kleene. Since $\mathbf{A}$ is standard, it is not even Kleene as a De Morgan lattice, i.e. there are $a, b \in \mathbf{A}$ such that $a \leq -a$, $b \leq -b$, and $a \nleq -b$. Let $c = a \vee (-a \wedge b)$ and $d = b \vee (-b \wedge a)$. Then $c = a \vee (-a \wedge b) \leq -a \wedge (a \vee -b) = -c$ and likewise $d \leq -d$. Moreover, $c \wedge -d = (a \vee (-a \wedge b)) \wedge -b \wedge (b \vee -a) \leq b \vee (-b \wedge a) = d$ and likewise $-c \wedge d \leq c$, $-c \leq c \vee -d$, and $-d \leq d \vee -c$. The De Morgan lattice $\{c \wedge d, c, d, c \vee d\}$ is therefore a subalgebra of the De Morgan lattice reduct of $\mathbf{A}$, let us call it $\mathbf{B}$. It is clear that $\{c \wedge d, c, d, c \vee d\} \subseteq \mathcal{I}_{\mathbf{A}}$, hence the algebra $\mathbf{B} \times \mathbf{B_2^s}$ has a subalgebra isomorphic to $\mathbf{DM_4^s} \times \mathbf{B_2^s}$, namely $\{c \wedge d, c, d, c \vee d\} \times \{\bot\} \cup \{-c \wedge -d, -c, -d, -c \vee -d\} \times \{\top\}$. It follows that $\mathsf{K} \subseteq \mathsf{RegNIDML}$.

To prove (vii), suppose that $\mathsf{K} \nsubseteq \mathsf{NIDML} \cup \mathsf{KL}$. Then there is some standard $\mathbf{A} \in \mathsf{K}$ which is neither Kleene nor non-idempotent. It is therefore neither Kleene nor non-idempotent as a De Morgan lattice, hence $\mathbf{DM_4^s} \subseteq \mathbf{A}$ by Lemma 2.4 and $\mathsf{TIL} \subseteq \mathsf{K}$. □

In the following proposition, by a *De Morgan $\mathcal{I}$-algebra* we shall mean a De Morgan $\mathcal{I}$-lattice with a bottom and top element which are part of the signature. The quasivariety of *Kleene $\mathcal{I}$-algebras*, i.e. De Morgan $\mathcal{I}$-algebras whose appropriate reduct is a Kleene $\mathcal{I}$-lattice, is then denoted KA.

Proposition 4.6. *Let K be a quasivariety of De Morgan $\mathcal{I}$-lattices ($\mathcal{I}$-algebras). Then each algebra in K is standard if and only if $\mathsf{K} \subseteq \mathsf{KL}$ ($\mathsf{K} \subseteq \mathsf{KA}$).*

Proof. Each Kleene $\mathcal{I}$-lattice is standard. Conversely, suppose that $\mathsf{K} \nsubseteq \mathsf{KL}$. Then there is some $\mathbf{A} \in \mathsf{K}$ with $a, b \in \mathcal{I}_{\mathbf{A}}$ such that $a \nleq -b$ in $\mathbf{A}$, hence $c \nleq -c$ for $c = a \vee b \in \mathcal{I}_{\mathbf{A}}$. But then either $-c < c$, in which case the non-standard expansion $\mathbf{B_2}$ is a subalgebra of $\mathbf{A}$, or $-c \nleq c$, in which case a non-standard expansion of $\mathbf{B_2} \times \mathbf{B_2}$ is a subalgebra of $\mathbf{A}$.

The proof for De Morgan $\mathcal{I}$-algebras is analogical, except instead of the non-standard expansions of $\mathbf{B_2}$ and $\mathbf{B_2} \times \mathbf{B_2}$ we may have to take the non-standard expansions of the extensions of these De Morgan lattices by an extra bottom and top element. □

Finally, we consider how the expressive power of our language changes when we replace the inconsistency predicate by an inconsistency constant. By a *De Morgan lattice with an inconsistency constant* (or briefly *De Morgan* 0*-lattice*) we shall mean a De Morgan lattice $\mathbf{A}$ equipped with a constant 0 which satisfies the equation $x \wedge -x \leq 0$.[2] Such an algebra is *standard* if $\mathcal{I}_{\mathbf{A}}$ is precisely the principal ideal generated by the element 0. Each finite De Morgan lattice has a unique expansion by a standard inconsistency constant,

[2] Do not confuse the constant 0 with the bottom element of the lattice.

Table 2. The quasivarieties of Kleene 0-lattices

Name	Abbreviation	Generated by
Boolean	BL	$\mathbf{B}_2^s$
Kleene	KL	$\mathbf{B}_2^s$, $\mathbf{K}_3^s$
regular Kleene	RegKL	$\mathbf{RegK}_4^s$
non-idempotent Kleene	NIKL	$\mathbf{K}_3^s \times \mathbf{B}_2^s$
idempotent Kleene	IdemKL	$\mathbf{K}_3^s$

namely $0 = \bigvee \mathcal{I}_{\mathbf{A}}$. A quasivariety of De Morgan 0-lattices is again called standard if it is generated by its standard elements.

Each inconsistency constant 0 on $\mathbf{A}$ defines the inconsistency predicate $\{a \in \mathbf{A} | a \leq 0\}$ on $\mathbf{A}$. All notions defined in the previous section of De Morgan $\mathcal{I}$-lattices thereby extend to De Morgan 0-lattices. In addition, we shall call a De Morgan 0-lattice *idempotent* if it satisfies the equation $-0 \leq 0$. We now show that idempotence is the only new property of *Kleene* lattices which may be expressed quasiequationally using the standard inconsistency constant (rather than the standard inconsistency predicate). By contrast, we show that there are infinitely many standard quasivarieties of De Morgan 0-lattices (although we do not establish exactly how many).

When talking about De Morgan 0-lattices, the notation $\mathbf{A}^s$ will be used to denote the expansion of the De Morgan lattice $\mathbf{A}$ by the standard inconsistency constant (provided that it exists), as in Table 2. Here by a *standard* inconsistency constant we mean one which generates a standard inconsistency predicate as its principal ideal. Note that the inconsistency constant 0 is standard if and only if there is a finite set $B \subseteq \mathbf{A}$ such that $0 = \bigvee_{a \in B}(a \wedge -a)$. It will also be useful to define 1 as -0.

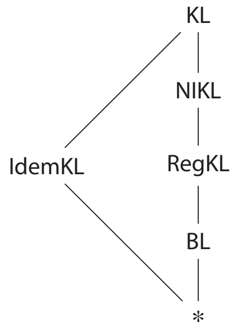
Theorem 4.7. *The quasivarieties of De Morgan* 0*-lattices introduced in the previous section are generated by the standard algebras shown in Table 2.*

Proof. Let $\mathbf{A}$ be a Kleene 0-lattice. Then the Kleene lattice reduct of $\mathbf{A}$ is a subdirect product of copies of $\mathbf{B}_2$ and $\mathbf{K}_3$. Since $0 \leq 1$, the component of 0 in each subdirect factor has to be the standard one on pain of violating the equality $0 \leq 1$, hence $\mathbf{A}$ is a subdirect product of $\mathbf{B}_2^s$ and $\mathbf{K}_3^s$. If $\mathbf{A}$ is Boolean, then it is Boolean as a Kleene lattice, hence it is a subdirect power of $\mathbf{B}_2^s$. If $\mathbf{A}$ is non-idempotent, i.e. if $1 \nleq 0$, then at least one of the subdirect factors has to be isomorphic to $\mathbf{B}_2^s$ (moreover, $\mathbf{B}_2^s \subseteq \mathbf{K}_3^s \times \mathbf{B}_2^s$). If on the other hand $\mathbf{A}$ is idempotent, i.e. if $1 = 0$, then all of the subdirect factors must be isomorphic to $\mathbf{K}_3^s$. Finally, if $\mathbf{A}$ is regular, i.e. if it satisfies the quasiequation $1 \wedge x \leq 0 \Rightarrow x \leq 0$, then $\mathbf{A}$ is regular as a Kleene lattice, therefore as a Kleene lattice it is a subdirect power of $\mathbf{RegK}_4$. But then the component of 0 in each subdirect factor again has to be the standard one, therefore $\mathbf{A}$ is a subdirect power of $\mathbf{RegK}_4^s$. □

We shall now describe the lattice of quasivarieties of Kleene 0-lattices. Note that all such quasivarieties are standard, since each Kleene 0-lattice is easily seen to be standard.

Theorem 4.8. *The lattice of quasivarieties of Kleene* 0*-lattices is the finite lattice shown in Figure 4.*

FIGURE 4. The quasivarieties of Kleene 0-lattices



Proof. If $\mathbf{A}$ is a non-trivial Kleene 0-lattice, then either $0 < 1$, in which case $\mathbf{B}_2^{\mathrm{s}} \subseteq \mathbf{A}$, or $0 = 1$, in which case $\mathbf{K}_3^{\mathrm{s}} \subseteq \mathbf{A}$. Therefore for each non-trivial quasivariety K of Kleene 0-lattices we have either $\mathsf{IdemKL} \subseteq \mathsf{K}$ or $\mathsf{BL} \subseteq \mathsf{K}$, and either $\mathsf{K} \subseteq \mathsf{IdemKL}$ or $\mathsf{BL} \subseteq \mathsf{K}$. Suppose therefore that $\mathsf{BL} \subseteq \mathsf{K}$, i.e. $\mathbf{B}_2^{\mathrm{s}} \in \mathsf{K}$. If $\mathsf{K} \nsubseteq \mathsf{BL}$, then there is some non-Boolean $\mathbf{A} \in \mathsf{K}$, hence $\mathbf{A} \times \mathbf{B}_2^{\mathrm{s}}$ is a non-idempotent non-Boolean Kleene 0-lattice and $\mathbf{RegK}_4^{\mathrm{s}} \subseteq \mathbf{A}$. If K is a quasivariety such that $\mathsf{K} \nsubseteq \mathsf{RegKL}$, then $\mathbf{B}_2^{\mathrm{s}} \in \mathsf{K}$ and there is some non-regular $\mathbf{A} \in \mathsf{K}$, i.e there is some $a \in \mathbf{A}$ such that $1 \wedge a \leq 0$ but $a \nleq 0$. Without loss of generality $a = a \vee 0$, i.e. $0 \leq a$. Then it is straightforward to verify that $\{a \wedge -a, 0, a, -a, 1, a \vee -a\}$ is a subalgebra of $\mathbf{A}$ isomorphic to $\mathbf{K}_3^{\mathrm{s}} \times \mathbf{B}_2^{\mathrm{s}}$. Finally, if $\mathsf{K} \nsubseteq \mathsf{NIKL}$, then there is some non-trivial $\mathbf{A} \in \mathsf{K}$ which is not non-idempotent, hence $\mathbf{K}_3^{\mathrm{s}} \subseteq \mathbf{A}$. □

In the following proposition, by a *De Morgan 0-algebra* we shall mean a De Morgan 0-lattice with a bottom and top element which are part of the signature. The quasivariety of *Kleene 0-algebras*, i.e. De Morgan 0-algebras whose appropriate reduct is a Kleene 0-lattice, is then denoted KA.

Proposition 4.9. *Let K be a quasivariety of De Morgan 0-lattices (0-algebras). Then each algebra in K is standard if and only if $\mathsf{K} \subseteq \mathsf{KL}$ ($\mathsf{K} \subseteq \mathsf{KA}$).*

Proof. Each Kleene 0-lattice is standard. Conversely, let K be a quasivariety of De Morgan 0-lattices such that $\mathsf{K} \nsubseteq \mathsf{KL}$. Then there is some $\mathbf{A} \in \mathsf{K}$ such that $0 \nleq 1$ in $\mathbf{A}$. But then either $1 < 0$, in which case the non-standard expansion of $\mathbf{B}_2$ is a subalgebra of $\mathbf{A}$, or $1 \nleq 0$, in which case a non-standard expansion of $\mathbf{B}_2 \times \mathbf{B}_2$ is a subalgebra of $\mathbf{A}$.

The proof for De Morgan 0-algebras is again analogical, except instead of the non-standard expansions of $\mathbf{B}_2$ and $\mathbf{B}_2 \times \mathbf{B}_2$ we may have to take the non-standard expansions of the extensions of these De Morgan lattices by an extra top and bottom element. □

We conclude this section by exhibiting an infinite decreasing chain of standard quasivarieties of (regular non-idempotent) De Morgan 0-lattices. Consider, for $n \geq 1$, the quasiequation

$$(\alpha_n) \qquad a_1 \leq -a_1, \dots, a_n \leq -a_n, a_1 \vee \dots \vee a_n = 0 \Rightarrow x = y.$$

This quasiequation states that 0 cannot be expressed as a disjunction of n or less elements of the form $a \wedge -a$. Clearly $\mathbf{A}$ is non-standard if and only if it satisfies (α_n) for each $n \geq 1$, hence the class of non-standard De Morgan 0-lattices is a quasivariety. The quasiequations (α_n) coincide with the quasiequations (β_n) used by Gaitán and Perea in [5] if we restrict to the variety of De Morgan 0-lattices defined by $x \leq 0$.

Lemma 4.10. *If $\mathbf{A}$ is a standard De Morgan 0-lattice which is generated as a De Morgan lattice by the finite set $X \subseteq \mathbf{A}$, then $0 = \bigvee\{x \wedge -x \mid x \in X\}$.*

Proof. It suffices to prove that for each term t and each tuple $\overline{x}$ of elements from X we have $t(\overline{x}) \wedge -t(\overline{x}) \leq \bigvee\{x \wedge -x \mid x \in X\}$. This is a straightforward proof by induction over the complexity of the term t. □

Let $\mathbf{A}_n$ be the free De Morgan lattice on n generators equipped with the standard inconsistency constant. Equivalently, $\mathbf{A}_n$ may be defined as the free distributive lattice on $2n$ generators of the form $x_i, -x_i$ for $1 \leq i \leq n$ with $0 = (x_1 \wedge -x_1) \vee \ldots \vee (x_n \wedge -x_n)$. For any generator x_i of $\mathbf{A}_n$, let $h_i : \mathbf{A}_n \to \mathbf{DM_4}$ denote the unique homomorphism of De Morgan lattices such that $h_i(i) = \mathsf{b}$ and $h_i(x_j) = \mathsf{n}$ for all generators x_j other than x_i, where b and n denote the two fixpoints of De Morgan negation on $\mathbf{DM_4}$.

Lemma 4.11. *The algebra $\mathbf{A}_n$ is regular and non-idempotent.*

Proof. The non-idempotence of $\mathbf{A}_n$ is witnessed by any h_i, as $h_i(1) = \bot \nleq \top = h_i(0)$. To prove that $\mathbf{A}_n$ is regular, suppose that $a \nleq 0$. Then there is a homomorphism $h : \mathbf{A}_n \to \mathbf{DM_4}$ such that $h(a) \nleq h(0)$. We wish to show that $h(1 \wedge a) \nleq h(0)$. Clearly $h(0) < \top$ and if $h(0) = \bot$, then $h(1 \wedge 0) = h(0) \nleq h(0)$, hence $1 \wedge a \nleq 0$. Suppose therefore without loss of generality that $h(0) = \mathsf{b}$. Since $h(0) = \mathsf{b}$, there is no x_i such that $h(x_i) = \mathsf{n}$. Therefore $h(a) = \top$ and there is some x_j such that $h(x_j) \in \{\bot, \top\}$, hence $h(1) = \top$ and $h(1 \wedge a) \nleq h(0)$. □

Lemma 4.12. *The algebra $\mathbf{A}_n$ satisfies (α_m) if and only if $m < n$.*

Proof. By Lemma 4.10, the inconsistency constant of $\mathbf{A}_n$ is a disjunction of n elements. Vice versa, suppose that in $\mathbf{A}_n$, the inconsistency constant is a disjunction of $a_i \wedge -a_i$ for $1 \leq i \leq m$. Then there is some generator x_j such that none of the elements a_i belong to the subalgebra generated by x. We wish to show that $x_j \wedge -x_j \nleq (a_1 \wedge -a_1) \vee \ldots \vee (a_m \wedge -a_m)$.

If the term t does not contain x_j, then $h_j(t) = \mathsf{n}$, and if the term t does not contain any variable other than x_j, then $h_j(t) = \mathsf{b}$. It follows that if the term t does not belong to the subalgebra generated by x_j, then $h_j(t) \neq \mathsf{b}$. Therefore, $h_j(x_j \wedge -x_j) \nleq h_j(a_1 \wedge -a_1) \vee \ldots \vee h_j(a_m \wedge -a_m)$. □

Proposition 4.13. *The class of all standard De Morgan $\mathcal{J}$-lattices (0-lattices) is not an elementary class.*

Proof. We show that these classes are not closed under ultraproducts. Let $\mathbf{A}$ be a non-principal ultraproduct of the standard algebras $\mathbf{A}_i$. Lemma 4.12 implies that $\mathbf{A}$ satisfies (α_n) for each $n \geq 1$. In other words, $\mathbf{A}$ is a non-standard De Morgan 0-lattice. The associated De Morgan $\mathcal{J}$-lattice, which is an ultraproduct of the standard De Morgan $\mathcal{J}$-lattices associated to the De Morgan 0-lattices $\mathbf{A}_i$, is therefore also non-standard. □

Theorem 4.14. *There is an infinite decreasing chain of standard quasivarieties of (regular non-idempotent) De Morgan* 0*-lattices.*

Proof. Take the quasivarieties generated by the sets of standard algebras $\{\mathbf{A}_n \mid n \geq i\}$ for $i \geq 1$. These generate standard quasivarieties of regular non-idempotent De Morgan 0-lattices. Moreover, these quasivarieties are distinct, as witnessed by the quasiequations (α_n). □

We do not know know many standard quasivarieties of De Morgan 0-lattices there are. Since totally inconsistent De Morgan 0-lattices are clearly termwise equivalent to De Morgan algebras, we know that there is a continuum of quasivarieties of De Morgan 0-lattices which satisfy the equation $x \wedge -x \leq y \vee -y$. However, no non-trivial *standard* De Morgan 0-lattice which satisfies this equation can be totally inconsistent, therefore this result tells us nothing about standard quasivarieties of De Morgan 0-lattices.

5. Conclusion

We have succeeded in the task of pinpointing just how much quasiequational expressive power the standard inconsistency predicate adds to De Morgan lattices: the only new properties which were not expressible quasiequationally in the language of De Morgan lattices are regularity, its conjunction and disjunction with non-idempotence, and total inconsistency.

We have also seen that the only quasiequational expressive gain of adding a standard inconsistency constant (as opposed to a standard inconsistency predicate) to Kleene lattices consists in being able to define the class of idempotent Kleene lattices quasiequationally.

However, it remains an open question to determine how many standard quasivarieties of De Morgan lattices with an inconsistency constant there are. We have only managed to show that there are infinitely many.

Another natural open question is the following: is the lattice of standard quasivarieties a sublattice of the lattice of all quasivarieties of De Morgan lattice with an inconsistency constant? Equivalently, is the intersection of two standard quasivarieties necessarily standard?

Acknowledgements

This research was supported by the research program Progres Q14 Rationality Crisis and Modern Thought. The author is grateful to the referees for their comments which helped to improve the paper.

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2017 ACTA UNIVERSITATIS CAROLINAE PAG. 63–72
PHILOSOPHICA ET HISTORICA 2 / MISCELLANEA LOGICA

GRIGORIEFF FORCING AND THE TREE PROPERTY

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ABSTRACT
In this paper we use Grigorieff forcing to obtain the tree property at the second successor of a regular uncountable cardinal κ. We also show that Silver forcing can be used to obtain the tree property at $\aleph_2$.
Keywords: Grigorieff forcing, Silver forcing, tree property
AMS subject code classification: 03E05

1. Introduction

Let μ be an infinite cardinal. We say that a tree T of height μ^+ is a μ^+-tree if its levels have size less than μ^+. A μ^+-tree T is *Aronszajn* if it has no cofinal branches; T is a *special Aronszajn tree* if there is a function f from T to μ which is injective on chains in T, i.e. if x, y in T are comparable, then $f(x) \neq f(y)$. We say that μ^+ has the *tree property* if there are no μ^+-Aronszajn trees. In 1930's, Nachman Aronszajn proved in ZFC that there is a special Aronszajn tree at ω_1. Therefore ω_1 does not have the tree property. In 1949, Ernst Specker [Spe49] generalized Aronszajn's original result by proving that if $\mu^{<\mu} = \mu$ then there exists a special Aronszajn tree at μ^+.[1] Hence to obtain the tree property at κ^{++}, we need to violate GCH at κ.

In 1972, William Mitchell (using ideas of Silver) proved in [Mit72] that the tree property at κ^{++}, where κ is regular, is consistent under the assumption of the existence of a weakly compact cardinal. He used a mixed support iteration of Cohen forcings; for details see [Mit72]. Later, James Baumgartner and Richard Laver showed in [BL79] that the tree property at ω_2 can be achieved by iterating Sacks forcing for ω up to a weakly compact cardinal. In 1980, Akihiro Kanamori generalized this result to an arbitrary κ^{++}, where κ is a regular cardinal, see [Kan80]. The proof is based on the fusion property of Sacks forcing.

In this paper, we use a suitably generalized Grigorieff forcing (and Silver forcing at ω) to achieve the same results (see Section 2 for definitions).

[1] Jensen [Jen72] proved that the existence of a special μ^+-Aronszajn tree is equivalent to the existence of a combinatorial object called the weak square ($\square^*_\mu$). $\square^*_\mu$ is strictly weaker than the assumption $\kappa^{<\kappa} = \kappa$.

https://doi.org/10.14712/24647055.2017.16

2. Grigorieff and Silver forcing

The forcing, which we now call Grigorieff forcing, was first defined by Grigorieff in [Gri71] for $\kappa = \omega$; its generalizations for uncountable cardinals were studied extensively, see for example [HV16] and [AG09]. In this paper we focus on Grigorieff forcing at uncountable regular cardinals; we also mention Silver forcing at ω which has many similarities with Grigorieff forcing. Note that Grigorieff forcing at ω is rather specific because it is defined with respect to an ideal which is not normal; we therefore choose to use at ω Silver forcing instead. In fact, a natural generalization of Silver forcing to uncountable cardinals leads to the definition of Grigorieff forcing; see Remark 2.4 for more details.

The following definition is taken from [HV16].

Definition 2.1. Let κ be a regular cardinal and let I be a subset of $\mathcal{P}(\kappa)$. We define $\mathbb{P}_I(\kappa, 1) = (P_I(\kappa, 1), \leq)$ as

(2.1) $$P_I(\kappa, 1) = \{f | f \text{ is a partial function from } \kappa \text{ to } 2 \text{ and } \mathrm{Dom}(f) \in I\},$$

Ordering is by reverse inclusion, i.e. for $p, q \in P_I(\kappa, 1)$, $p \leq q$ if and only if $q \subseteq p$.

By varying I, we get Cohen forcing[2], Silver forcing and Grigorieff forcing. If I is the ideal of bounded subsets, then $\mathbb{P}_I(\kappa, 1)$ is the usual Cohen forcing $\mathrm{Add}(\kappa, 1)$. If I is a set of "coinfinite" subsets of ω, i.e. $I = \{x \subset \omega || \omega \setminus x| = \omega\}$, then we get Silver forcing at ω. If I is an arbitrary ideal on κ, then we obtain the definition of Grigorieff forcing at κ.

Definition 2.2. Let κ be a regular cardinal and let I be an ideal on κ. We define κ-Grigorieff forcing as $\mathbb{G}_I(\kappa, 1) = \mathbb{P}_I(\kappa, 1)$.

Definition 2.3. Let $I = \{x \subset \omega || \omega \setminus x| = \omega\}$. We define Silver forcing as $\mathbb{S}(\omega, 1) = \mathbb{P}_I(\omega, 1)$.

Remark 2.4. In principle, one can consider the following generalizations of Silver forcing at an uncountable cardinal κ. Consider $\mathbb{P}_{I_i}(\kappa, 1)$, $i < 3$, where: $I_0 = \{x \subset \kappa || \kappa \setminus x| = \kappa\}$, $I_1 = \{x \subset \kappa | \kappa \setminus x \text{ is stationary}\}$ and $I_2 = \{x \subset \kappa | \kappa \setminus x \text{ is closed unbounded}\}$. It is easy to see that I_0 and I_1 give rise to forcing notions which are not even ω_1-closed, and tend to collapse cardinals; I_2 behaves reasonably and in fact it is Grigorieff forcing with the non-stationary ideal. The definition with I_0 is only suitable for ω.

Now we discuss the basic properties of these forcings, in particular the chain condition and the closure.

Definition 2.5. Let $\mathbb{P}$ be a forcing notion and κ a regular infinite cardinal. We say that $\mathbb{P}$ is:

- κ-cc if every antichain of $\mathbb{P}$ has size less than κ.
- κ-Knaster if for every $X \subseteq \mathbb{P}$ with $|X| = \kappa$ there is $Y \subseteq X$, such that $|Y| = \kappa$ and all elements of Y are pairwise compatible.
- κ-closed if every decreasing sequence of conditions in $\mathbb{P}$ of size less than κ has a lower bound.

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2 The Cohen forcing for adding a new subset of a regular cardinal κ is composed of function from κ to 2 of size less than κ with the reverse inclusion ordering. We denote the Cohen forcing as $\mathrm{Add}(\kappa, 1)$.

Lemma 2.6. *Assume $2^\kappa = \kappa^+$. Then the forcing $\mathbb{P}_I(\kappa, 1)$ is κ^{++}-cc.*

Proof. This is easy observation about the size of the forcing. If $2^\kappa = \kappa^+$, then $|\mathbb{P}_I(\kappa, 1)| = \kappa^+$. Therefore $\mathbb{P}_I(\kappa, 1)$ is κ^{++}-cc. □

The properties of Grigorieff forcing depend on the properties of the given ideal. Recall the following definitions for a regular cardinal κ.

Definition 2.7. We say that an ideal I on κ is *κ-complete* if it is closed under the unions of less than κ-many elements of I.

Definition 2.8. We say that an ideal I on κ is *normal* if it is closed under the diagonal unions of κ-many elements of I, where the diagonal union for a sequence $\langle X_\alpha \subseteq \kappa | \alpha < \kappa\rangle$ of subsets of κ is defined as follows:

$$\Sigma_{\alpha<\kappa} X_\alpha = \{\xi < \kappa | \xi \in \bigcup_{\beta<\xi} X_\beta\} \tag{2.2}$$

Lemma 2.9. *Let κ be an uncountable regular cardinal and I be a κ-complete ideal on κ. If $\alpha < \kappa$ and $\langle p_\beta | \beta < \alpha\rangle$ is a decreasing sequence in $\mathbb{G}_I(\kappa, 1)$, then $p = \bigcup_{\beta<\alpha} p_\beta \in \mathbb{G}_I(\kappa, 1)$. Therefore $\mathbb{G}_I(\kappa, 1)$ is κ-closed.*

Proof. The proof is a direct consequence of the assumption that I is a κ-complete ideal. □

By the previous results, if I is a κ-complete ideal on an uncountable regular κ and $2^\kappa = \kappa^+$ then all cardinals greater than κ^+ and all cardinals less than or equal κ are preserved by Grigorieff forcing at κ. Also if CH holds then Silver forcing preserves all cardinals greater than ω_1.

To show that κ^+ and ω_1 are also preserved by Grigorieff forcing and Silver forcing, respectively, we need to introduced the concept of a fusion sequence.

2.1 Grigorieff forcing

For the rest of the section assume that κ is an uncountable regular cardinal.

Definition 2.10. For $\alpha < \kappa$ and $p, q \in \mathbb{G}_I(\kappa, 1)$ we define

$$p \leq_\alpha q \Leftrightarrow p \leq q \text{ and } \mathrm{Dom}(p) \cap (\alpha + 1) = \mathrm{Dom}(q) \cap (\alpha + 1). \tag{2.3}$$

We say that $\langle p_\alpha | \alpha < \kappa\rangle$ is a *fusion sequence* if for every α, $p_{\alpha+1} \leq_\alpha p_\alpha$ and $p_\beta = \bigcup_{\alpha<\beta} p_\alpha$ for every limit $\beta < \kappa$.

Lemma 2.11. *Let I be a normal ideal on κ. If $\langle p_\alpha | \alpha < \kappa\rangle$ is a fusion sequence in $\mathbb{G}_I(\kappa, 1)$, then the union $p = \bigcup_{\alpha<\kappa} p_\alpha$ is a condition in $\mathbb{G}_I(\kappa, 1)$ and $p \leq_\alpha p_\alpha$ for each $\alpha < \kappa$.*

Proof. It is sufficient to show that $\bigcup_{\alpha<\kappa} \mathrm{Dom}(p_\alpha)$ is in I, or equivalently $\bigcap_{\alpha<\kappa}(\kappa \setminus \mathrm{Dom}(p_\alpha))$ is in I^*, where I^* is the dual filter for I. Since I^* is a normal filter, the diagonal intersection $\triangle_{\alpha<\kappa}(\kappa \setminus \mathrm{Dom}(p_\alpha)) = \{\xi < \kappa | \xi \in \bigcap_{\beta<\xi}(\kappa \setminus \mathrm{Dom}(p_\beta))\}$ is in I^* and also the set $\{\beta < \kappa | \beta \text{ is a limit ordinal}\}$ is in I^* since I extends the nonstationary ideal on κ.

To finish the proof, it is enough to show that

$$\{\beta < \kappa | \beta \text{ is a limit ordinal}\} \cap \triangle_{\alpha<\kappa}(\kappa \setminus \mathrm{Dom}(p_\alpha)) \subseteq \bigcap_{\alpha<\kappa}(\kappa \setminus \mathrm{Dom}(p_\alpha)). \tag{2.4}$$

Let β be a limit ordinal in $\triangle_{\alpha<\kappa}(\kappa \setminus \mathrm{Dom}(p_\alpha))$. Then for all $\gamma < \beta$, $\beta \notin \mathrm{Dom}(p_\gamma)$. By the limit step of the definition of fusion sequence, $\beta \notin \mathrm{Dom}(p_\beta)$. Hence β is not in $\mathrm{Dom}(p_\alpha)$ for each $\alpha > \beta$ by (2.3). Therefore β is in $\bigcap_{\alpha<\kappa}(\kappa \setminus \mathrm{Dom}(p_\alpha))$. ☐

Corollary 2.12. *Let κ be an uncountable cardinal. Assume that $\kappa^{<\kappa} = \kappa$ and I is a normal ideal on κ. Then $\mathbb{G}_I(\kappa, 1)$ preserves κ^+.*

Remark 2.13. The proof of the previous corollary is a standard argument using the closure of the forcing under the fusion sequences. If $\dot{f}$ is a $\mathbb{G}_I(\kappa, 1)$-name for a function from κ to κ^+ then we construct by induction a fusion sequence such that its lower bound will force $\dot{f}$ is bounded. For the details for an inaccessible κ see Theorem 2.6 in [HV16]. If κ is a successor cardinal, a diamond-guided construction is usually invoked since it can show the preservation of κ^+ even for iterations of Grigorieff forcing (see section 2.3). However, it is easy to use a diagonal argument to show that $\mathbb{G}_I(\kappa, 1)$ preserves κ^+ even without the diamond (since $\kappa^{<\kappa} = \kappa$ implies the diamond at κ for all κ except ω_1, this observation is relevant only for $\mathbb{G}_I(\omega_1, 1)$).

Remark 2.14. The converse direction holds as well. For the proof see [HV16].

Remark 2.15. It is instructive to see the importance of having $(\alpha + 1)$ and not just α in (2.3). If we required that the domains are the same on α only, it is easy to construct a fusion sequence without a lower bound.[3]

2.2 Silver forcing

The fusion argument for Grigorieff forcing at ω is more complicated since at ω we do not have the notion of a normal ideal. For more details about the case of ω, see [Gri71]. For Silver forcing, a fusion sequence can be defined as follows:

Definition 2.16. If p is a condition in $\mathbb{S}(\omega, 1)$, let n_p denote the n-th element of $\omega \setminus \mathrm{Dom}(p)$. For $n < \omega$ and $p, q \in \mathbb{S}(\omega, 1)$ we define

$$p \leq_n q \Leftrightarrow p \leq q \text{ and } \mathrm{Dom}(p) \cap (n_q + 1) = \mathrm{Dom}(q) \cap (n_q + 1). \tag{2.5}$$

We say that $\langle p_n | n < \kappa \rangle$ is a *fusion sequence* if for every n, $p_{n+1} \leq_n p_n$.

Lemma 2.17. *If $\langle p_n | n < \omega \rangle$ is a fusion sequence in $\mathbb{S}(\omega, 1)$, then the union $p = \bigcup_{n<\omega} p_n$ is a condition in $\mathbb{S}(\omega, 1)$ and $p \leq_n p_n$ for each $n < \omega$.*

Proof. The proof follows from (2.5) since at the n-th step we guaranteed that n_{p_n} is not in $\mathrm{Dom}(p)$. ☐

Corollary 2.18. *ω_1 is preserved by Silver forcing.*

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[3] For instance consider the sequence $\langle p_\alpha | \alpha < \kappa \rangle$ of functions, where $\mathrm{Dom}(p_\alpha)$ is α for every $\alpha < \kappa$. If we changed the definition in (2.3) to require that the domains are equal on α only, then this is a fusion sequence without a lower bound (its greatest lower bound is a function with the domain equal to κ).

For the rest of the section, we fix an uncountable regular cardinal κ and a normal ideal I on κ. We will consider the iteration of Grigorieff forcing defined with respect to κ and I (for more details about iterations in general, see [Bau83]).

Definition 2.19. Let $\lambda > 0$ be an ordinal. Then we define $\mathbb{G}_I(\kappa, \lambda)$ by induction as follows:

(i) The forcing $\mathbb{G}_I(\kappa, 1)$ is defined as in Definition 2.2.

(ii) $\mathbb{G}_I(\kappa, \xi + 1) = \mathbb{G}_I(\kappa, \xi) * \dot{Q}_\xi$, where $\dot{Q}_\xi$ is a $\mathbb{G}_I(\kappa, \xi)$-name for the partial order $\mathbb{G}_I(\kappa, 1)$ as defined in the extension $V[\mathbb{G}_I(\kappa, \xi)]$.

(iii) For a limit ordinal ξ, $\mathbb{G}_I(\kappa, \xi)$ is the inverse limit of $\langle \mathbb{G}_I(\kappa, \zeta) | \zeta < \xi \rangle$ if $\mathrm{cf}(\xi) \leq \kappa$ and the direct limit otherwise.

We consider $\mathbb{G}_I(\kappa, \lambda)$ as the collection of functions p with domain λ such that for every $\xi < \lambda$, $p \restriction \xi \Vdash_\xi p(\xi) \in \dot{Q}_\xi$ and $|\mathrm{supp}(p)| \leq \kappa$. The ordering is defined as follows: for p, q in $\mathbb{G}_I(\kappa, \lambda)$, $p \leq q$ if and only if $\mathrm{supp}(p) \supseteq \mathrm{supp}(q)$ and for every $\xi \in \mathrm{supp}(q)$, $p \restriction \xi \Vdash_\xi p(\xi) \leq q(\xi)$.

Lemma 2.20. *Let κ be a regular cardinal and $\lambda > \kappa$ be an inaccessible cardinal. Then $\mathbb{G}_I(\kappa, \lambda)$ has size λ and it is λ-Knaster.*

Proof. See Theorem 16.30 in [Jec03]. Theorem 16.30 is formulated for the chain condition, but it is easy to check that the reformulation of the proof for Knaster forcings actually gives Knasterness. □

The following definitions and results are analogues of the corresponding results in [Kan80] which deals with Sacks forcing. We define the notion of meet and use it to show that the iteration of Grigorieff forcing is sufficiently closed and has the fusion property.

Definition 2.21. Let α be an ordinal. If $\langle p_\beta | \beta < \alpha \rangle$ is a decreasing sequence of conditions, then the *meet* $p = \bigwedge_{\beta<\alpha} p_\beta$ is defined as follows:

$$\mathrm{supp}(p) = \bigcup_{\beta<\alpha} \mathrm{supp}(p_\beta) \text{ and } p \restriction \gamma \Vdash p(\gamma) = \bigcup_{\beta<\alpha} p_\beta(\gamma) \text{ for } \gamma \in \mathrm{supp}(p). \tag{2.6}$$

Lemma 2.22. *If $\alpha < \kappa$ and $\langle p_\beta | \beta < \alpha \rangle$ is a decreasing sequence in $\mathbb{G}_I(\kappa, \lambda)$, then $p = \bigwedge_{\beta<\alpha} p_\beta \in \mathbb{G}_I(\kappa, \lambda)$. Hence $\mathbb{G}_I(\kappa, \lambda)$ is κ-closed.*

Proof. See Theorem 2.5 in [Bau83]. □

Definition 2.23. Let $p, q \in \mathbb{G}_I(\kappa, \lambda)$, $X \subseteq \lambda$ with $|X| < \kappa$ and $\alpha < \kappa$. We define

$$p \leq_{X,\alpha} q \Leftrightarrow p \leq q \text{ and } p \restriction \xi \Vdash p(\xi) \leq_\alpha q(\xi) \text{ for all } \xi \in X. \tag{2.7}$$

We say that a pair $(\langle p_\xi | \xi < \kappa \rangle, \langle X_\xi | \xi < \kappa \rangle)$ is a *fusion sequence* if it satisfies the following conditions:

(i) $p_{\xi+1} \leq_{X_\xi,\xi} p_\xi$ for every $\xi < \kappa$ and $p_\zeta = \bigwedge_{\xi<\zeta} p_\xi$ for every limit $\zeta < \kappa$;

(ii) $|X_\xi| < \kappa$ and $X_\xi \subseteq X_{\xi+1}$ for every $\xi < \kappa$;

(iii) $X_\zeta = \bigcup_{\xi<\zeta} X_\xi$ for every limit $\zeta < \kappa$ and $\bigcup_{\xi<\kappa} X_\xi = \bigcup_{\xi<\kappa} \mathrm{supp}(p_\xi)$.

Lemma 2.24. *Let $\lambda > 0$ be an ordinal. If $(\langle p_\beta | \beta < \kappa \rangle, \langle X_\beta | \beta < \kappa \rangle)$ is a fusion sequence, then $p = \bigwedge_{\beta<\kappa} p_\beta$ is in $\mathbb{G}_I(\kappa, \lambda)$.*

Proof. We prove the lemma by induction on $\xi \le \lambda$ and we show that for each $\xi \le \lambda$, $p \restriction \xi \in \mathbb{G}_I(\kappa, \xi)$.

If $\xi = 0$, then $p(\xi)$ is in $\mathbb{G}_I(\kappa, 1)$ by Lemma 2.11.

If $\xi = \zeta + 1$, then we want to show that $p \restriction \zeta \Vdash_\zeta p(\zeta) \in \dot{Q}_\zeta$. Since $p \restriction \zeta \le p_\beta \restriction \zeta$ for all $\beta < \kappa$, it is clear that $p \restriction \zeta \Vdash_\zeta$ "$\langle p_\beta(\zeta) | \beta < \kappa \rangle$ is a decreasing sequence in $\dot{Q}_\zeta$".

If ζ is not in $\mathrm{supp}(p)$, then we are done, since $p \restriction \zeta \Vdash_\zeta p(\zeta) = \check{1} \in \dot{Q}_\zeta$.

If $\zeta \in \bigcup_{\xi<\kappa} \mathrm{supp}(p_\xi)$, then by the definition of meet, we know that $p \restriction \zeta \Vdash p(\zeta) = \bigcup_{\beta<\kappa} p_\beta(\zeta)$. Now we use the properties of fusion sequence to show $p \restriction \zeta \Vdash \bigcup_{\beta<\kappa} p_\beta(\zeta) \in \dot{Q}_\zeta$. Since $\bigcup_{\beta<\kappa} X_\beta = \bigcup_{\beta<\kappa} \mathrm{supp}(p_\beta)$, there is $\alpha < \kappa$ and X_α such that $\zeta \in X_\alpha$. As the sequence $\langle X_\beta | \beta < \kappa \rangle$ is increasing and $p \restriction \zeta \le p_\beta \restriction \zeta$ for all $\beta < \kappa$, we have that $p \restriction \zeta \Vdash p_{\beta+1}(\zeta) \le_\beta p_\beta(\zeta)$ for all $\alpha \le \beta < \kappa$. Therefore $p \restriction \zeta \Vdash \bigcup_{\alpha\le\beta<\kappa} p_\beta(\zeta) \in \dot{Q}_\zeta$ by Lemma 2.11. Since $p \restriction \zeta \Vdash_\zeta$ "$\langle p_\beta(\zeta) | \beta < \kappa \rangle$ is a decreasing sequence in $\dot{Q}_\zeta$", $p \restriction \zeta \Vdash \bigcup_{\alpha\le\beta<\kappa} p_\beta(\zeta) = \bigcup_{\beta<\kappa} p_\beta(\zeta) \in \dot{Q}_\zeta$.

If ξ is a limit ordinal, then the claim is clear. □

The fusion property is used to show that κ^+ is preserved in the extension by $\mathbb{G}_I(\kappa, \lambda)$.

Fact 2.25. *Assume that either κ is inaccessible or that $\diamondsuit_\kappa$ holds. Then $\mathbb{G}_I(\kappa, \lambda)$ preserves κ^+.*

Proof. Follows from [Kan80] by adapting the argument with the fusion defined for Grigorieff forcing. □

3. Forcing the tree property

In this section, let us assume that κ is an uncountable regular cardinal and I is a normal ideal on κ.

3.1 Fusion and not adding branches

This section is based on the paper [FH15] where a general notion of fusion was defined. Both Grigorieff and Silver forcing satisfy this general notion, and we can therefore use a criterion from [FH15] to argue that new branches are not added to certain trees. To prove Fact 3.6, we need to apply the criterion to the iteration $\mathbb{G}_I(\kappa, \lambda)$ for an arbitrary uncountable regular κ. To illustrate the method, we will assume that κ is inaccessible and the iteration has length 1. Longer iterations for an inaccessible κ are more complicated notationally, but do not introduce new ideas. If κ is a successor cardinal, a diamond-guided construction must be used.

Definition 3.1. Let $\mathbb{P}$ be a forcing notion and G a $\mathbb{P}$-generic filter. We say that a sequence of ground-model objects $x = \langle a_i | i < \kappa \rangle$ in $V[G]$ is *fresh* if for every $\alpha < \kappa$, $x \restriction \alpha$ is in V, but x is in $V[G] \setminus V$.

Lemma 3.2. *Let $\mathbb{P}$ be a forcing notion and let the weakest condition of $\mathbb{P}$ force that $\dot{f}$ is a fresh κ-sequence. Then for every p_0 and p_1 in $\mathbb{P}$ and every $\delta < \kappa$ there are $r_0 \leq p_0$, $r_1 \leq p_1$ and $\gamma \geq \delta$ such that r_0 and r_1 force contradictory information about $\dot{f}$ at level γ.*

Proof. Let p_0, p_1 and $\delta < \kappa$ be given. Since $\dot{f}$ is a fresh sequence there are $q^0, q^1 < p_0$ and $\gamma > \delta$ such that q^0 and q^1 force contradictory information about $\dot{f}$ at γ. Also there is $r_1 \leq p_1$ which decides the value of $\dot{f}$ at γ to be some element of the ground model a. Since q^0 and q^1 force contradictory information about $\dot{f}$ at γ, at least one of them has to force $\dot{f}(\gamma) \neq a$. Chose r_0 to be the one with smaller upper index which forces this. □

Definition 3.3. Assume $\kappa^{<\kappa} = \kappa$. We say that $\mathbb{G}_I(\kappa, 1)$ *does not decide fresh κ^+-sequences in a strong sense* if the following hold: whenever $\dot{f}$ is a name for a fresh sequence of length κ^+, i.e

$$\mathbb{G}_I(\kappa,1) \Vdash \text{“} \dot{f} \text{ is a name for a fresh sequence of length } \kappa^+, \text{”} \tag{3.1}$$

then for every $p \in \mathbb{G}_I(\kappa, 1)$, every $\alpha < \kappa$ and every $\delta < \kappa^+$, there are $p_0 \leq_\alpha p$ and $p_1 \leq_\alpha p$ and γ, with $\delta < \gamma < \kappa^+$, such that whenever $r_0 \leq p_0$ and $r_1 \leq p_1$ and

$$r_0 \Vdash \dot{f} \restriction \gamma = \check{f}_0 \text{ and } r_1 \Vdash \dot{f} \restriction \gamma = \check{f}_1 \tag{3.2}$$

Then

$$f_0 \neq f_1. \tag{3.3}$$

That means, r_0 and r_1 force contradictory information about $\dot{f}$ restricted to γ.

Theorem 3.4. *Let κ be an inaccessible cardinal. If $\mu \geq \kappa$ is such that $2^\kappa > \mu$, then $\mathbb{G}_I(\kappa, 1)$ does not add cofinal branches to μ^+-trees.*

Proof. We use Theorem 3.4 from [FH15], which says that it is enough to verify that Grigorieff forcing $\mathbb{G}_I(\kappa, 1)$ does not decide κ^+-sequence in a strong sense.

Assume that $1 \Vdash$“ $\dot{b}$ is a fresh sequence of length κ^+”. Now we need to show that for any $\alpha < \kappa$, $\delta < \kappa^+$, and condition p, there are conditions p_0, p_1 and ordinal γ such that $p_0 \leq_\alpha p, p_1 \leq_\alpha p, \delta < \gamma < \kappa^+$ and whenever $r_0 \leq p_0$ and $r_1 \leq p_1$ are such that

$$r_0 \Vdash \dot{b} \restriction \gamma = \check{b}_0 \text{ and } r_1 \Vdash \dot{b} \restriction \gamma = \check{b}_1. \tag{3.4}$$

Then

$$b_0 \neq b_1. \tag{3.5}$$

Denote $A = \{(f,g) | f, g \in {}^{\alpha+1}2 \text{ and } f \leq p \restriction \alpha+1 \text{ and } g \leq p \restriction \alpha+1\}$. Since κ is inaccessible, the size of A is less than κ.

We will construct by induction on $|A|$ two $\leq_\alpha$-decreasing sequences continuous at limits $\langle p_0^i | i < |A| \rangle$ and $\langle p_1^i | i < |A| \rangle$ which satisfy

$$p_0^i \restriction \alpha+1 = p_1^i \restriction \alpha+1 = p \restriction \alpha+1 \tag{3.6}$$

for all $i < |A|$; p_0 will be the infimum of $\langle p_0^i | i < |A| \rangle$ and p_1 the infimum of $\langle p_1^i | i < |A| \rangle$. We will also construct an increasing sequence of ordinals continuous at limits $\langle \gamma_i | i < |A| \rangle$. The desired γ will be the supremum of this sequence. Enumerate $A = \{(f,g)_i | i < |A|\}$.

Set $p_0^0 = p$ and $p_1^0 = p$ and $\gamma_0 > \delta$.

For $m < |A|$, assume p_j^m, for $j \in \{0,1\}$, and γ_m were already constructed. To construct the $m+1$-st element of the sequences, and also γ_{m+1}, consider $(f,g) = (f,g)_m$.

Consider the conditions $p_0^m \cup f$ and $p_1^m \cup g$. By Lemma 3.2, find $s_0 \leq p_0^m \cup f$ and $s_1 \leq p_1^m \cup g$ such that s_0 and s_1 force contradictory information about $\dot{b}$ at level β for some $\beta > \gamma_m$. Set p_0^{m+1} to be $p_0^m \cup s_0 \restriction [\alpha+1, \kappa)$ and p_1^{m+1} to be $p_1^m \cup s_1 \restriction [\alpha+1,\kappa)$ and $\gamma_{m+1} = \beta$.

At limit stages, take the infimum of the conditions and the supremum of the ordinals.

We now verify that $p_0 = \bigwedge \langle p_0^i | i < |A| \rangle$, $p_1 = \bigwedge \langle p_1^i | i < |A| \rangle$, and $\gamma = \sup \langle \gamma_i | i < |A| \rangle$ are as desired. Let $r_0 \leq p_0$ and $r_1 \leq p_1$ be given. We can assume that both r_0 and r_1 are defined on $\alpha+1$. Then there is some $(f,g)_m \in A$ such that $r_0 \leq p_0^{m+1} \cup f$ and $r_1 \leq p_1^{m+1} \cup g$, and so r_0 and r_1 decide $\dot{b}$ differently at $\gamma_{m+1} < \gamma$. □

Remark 3.5. Note that the previous proof can be easily modified for Silver forcing at ω and its definition of fusion.

Fact 3.6. *Assume that either κ is inaccessible or that $\diamondsuit_\kappa$ holds. Let $\lambda > 0$ be an ordinal. If $\mu \geq \kappa$ is such that $2^\kappa > \mu$, then $\mathbb{G}_I(\kappa,\lambda)$ does not add cofinal branches to μ^+-trees.*

Remark 3.7. Note that for $\kappa = \xi^+ > \omega_1$, we just need to assume $2^\xi = \xi^+$, since this ensures $\diamondsuit_\kappa$.

3.2 The tree property

We showed in the previous section that under GCH, $\mathbb{G}_I(\kappa,\lambda)$ preserves all cardinals smaller or equal to κ (by κ-closure) and cardinals greater or equal to λ (by λ-cc). Moreover, under an additional assumption, κ^+ is preserved due to the fusion property.

Now we show that cardinals in the interval (κ^+, λ) are collapsed.

Lemma 3.8. *Assume that either κ is inaccessible or that $\diamondsuit_\kappa$ holds. Let $\lambda > \kappa$ be an inaccessible cardinal. Then $V[\mathbb{G}_I(\kappa,\lambda)] \models \lambda = \kappa^{++}$.*

Proof. The preservation of κ^+ follows by Fact 2.25, and the collapse of λ to become the second successor of κ follows by the more general fact which says that Cohen forcing at κ^+ is regularly embedded to any κ-support iteration of non-trivial forcing notions of length (at least) κ^+. □

Now we have everything that we need to prove the main theorem of this paper.

Theorem 3.9. *Assume GCH. Assume κ is regular uncountable. If there exists a weakly compact cardinal $\lambda > \kappa$, then in the generic extension by $\mathbb{G}_I(\kappa,\lambda)$, the following hold:*

(i) $2^\kappa = \lambda = \kappa^{++}$;

(ii) κ^{++} has the tree property.

Proof. For simplicity, we assume that λ is measurable.[4]

[4] If λ is just a weakly compact cardinal, we modify the argument as follows. If $\dot{T}$ is a nice name for a λ-tree, fix $j : M \to N$ so that M is a transitive model of ZFC$^-$ of size λ closed under $<\lambda$-sequences which contains as elements the forcing $\mathbb{G}_I(\kappa,\lambda)$ and $\dot{T}$, j has critical point λ, N has size λ, is closed under $<\lambda$-sequences and $M \in N$ (in particular, $\dot{T}$ is in N). The existence of such j follows from the weak compactness of λ. Then apply the argument below to this j.

Ad (i). It is easy to see that $2^\kappa = \lambda$ and $\lambda = \kappa^{++}$ follows from Lemma 3.8.

Ad (ii). Let G be a $\mathbb{G}_I(\kappa, \lambda)$-generic filter over V. Since λ is measurable in V, there is an elementary embedding $j : V \to M$ with critical point λ and ${}^{\lambda}M \subseteq M$, where M is a transitive model of ZFC.

In M, the forcing $j(\mathbb{G}_I(\kappa, \lambda))$ is the iteration of $\mathbb{G}_I(\kappa, 1)$ of length $j(\lambda)$ with κ-support by the elementarity of j. The forcing $\mathbb{G}_I(\kappa, j(\lambda))^M$ is forcing equivalent to $(\mathbb{G}_I(\kappa, \lambda) * \dot{\mathbb{G}}_I(\kappa, [\lambda, j(\lambda)))^M$. As j is the identity below λ, $\mathbb{G}_I(\kappa, \alpha) = \mathbb{G}_I(\kappa, \alpha)^M$, for $\alpha < \lambda$ and since we take direct limit at λ, $\mathbb{G}_I(\kappa, \lambda) = \mathbb{G}_I(\kappa, \lambda)^M$. Hence G is also $\mathbb{G}_I(\kappa, \lambda)^M$-generic over M.

Let H be $\mathbb{G}_I(\kappa, [\lambda, j(\lambda)))^{M[G]}$-generic over $V[G]$, and let us work in $V[G][H]$. Since we have $j[G] \subseteq G * H$, we can use Silver lifting lemma (see Proposition 9.1 in [Cum10]) and lift j to $j^* : V[G] \to M[G][H]$.

Assume T is a λ-tree in $V[G]$; we show that T has a cofinal branch in $V[G]$, and therefore there is no λ-Aronszajn tree in $V[G]$.

We can consider T as a subset of λ. Let $\dot{T}$ be a nice name for T in V. As $\dot{T}$ is an element of $H(\lambda^+)$, $\dot{T}$ is in M, and hence T is in $M[G]$. By elementarity of j^*, $j^*(T)$ is a $j^*(\lambda)$-tree in $M[G][H]$, hence it has a node b of length λ in $M[G][H]$. As j^* is the identity below λ, $j^*(T) \restriction \lambda = T$; therefore b is a cofinal branch trough T in $M[G][H]$.

By Fact 3.6, $\mathbb{G}_I(\kappa, [\lambda, j(\lambda)))^{M[G]}$ does not add cofinal branches to λ-trees over $M[G]$. Therefore b is in $M[G]$, and hence in $V[G]$. □

Remark 3.10. As we noted above (see Remark 3.5), the Silver forcing at ω satisfies the criterion for not adding branches from [FH15]; therefore it is easy to show (as in Theorem 3.9) that $\mathbb{S}(\omega, \lambda)$ forces the tree property at ω_2 if λ is a weakly compact cardinal.

Remark 3.11. We say that an uncountable μ^+ has the *weak tree property* if there are no special μ^+-Aronszajn trees. One can show that whenever GCH holds and κ is regular, $\mathbb{G}_I(\kappa, \lambda)$ and $\mathbb{S}(\omega, \lambda)$ force the weak tree property at κ^{++} and $\aleph_2$, respectively, whenever λ is a Mahlo cardinal greater than κ. The proof is a variant of the argument in Theorem 3.8; for more details, see [Mit72].

3.3 Open question

Q1. As in [Ung12], one may ask about the indestructibility of the tree property in the models obtained by Silver and Grigorieff forcing. For instance, one can ask: Is the tree property at κ^{++} obtained by Grigorieff forcing indestructible under Cohen forcing at κ?

Q2. Or more generally, one may study the indestructibility over models with the tree property obtained by forcings which satisfy some kind of fusion (Sacks, Grigorieff, Silver, axiom-A forcing notions, etc.).

Acknowledgments

The author was supported by FWF/GAČR grants I 1921-N25 and 17-33849L.

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ACTA UNIVERSITATIS CAROLINAE
PHILOSOPHICA ET HISTORICA 2/2017

MISCELLANEA LOGICA

Editor: Michal Peliš
Cover and layout Kateřina Řezáčová
Published by Charles University
Karolinum Press, Ovocný trh 560/5, 116 36 Prague 1, Czech Republic
www.karolinum.cz
Prague 2017
Typeset by Lacerta (www.sazba.cz)
Printed by Karolinum Press
ISSN 0567-8293 (Print)
ISSN 2464-7055 (Online)
MK ČR E 18598

Distributed by Faculty of Arts, Charles University, nám. Jana Palacha 2,
116 38 Praha 1, Czech Republic (books@ff.cuni.cz)