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COMPLETENESS OF PREDICATE GENTZEN CALCULUS WITH RESPECT TO INTUITIONISTIC KRIPKE SEMANTICS

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ABSTRACT
This paper deals with intuitionistic logic and completeness of Gentzen calculus with respect to its semantics. We present a rather simple proof for the case where the language is at most countable but may contain function symbols.
Keywords: completeness, Gentzen calculus, intuitionistic predicate logic, tree proof, saturated sequent.

1. Introduction

Completeness theorems for intuitionistic predicate logic with respect to Kripke semantics are often formulated with an additional assumption that the language is at most countable or that it contains no function symbols (or both). For example, van Dalen's chapter in the Handbook of Philosophical Logic [2] contains a completeness proof of Hilbert calculus where both the additional conditions are in place. Takeuti's book [4] contains a proof for Gentzen calculus for the case where the language has arbitrary cardinality, but the absence of function symbols is still assumed.

We will show a completeness proof for the Gentzen calculus which could be seen as complementary to Takeuti's. We assume that the language is at most countable, but we do admit function symbols. Our proof is inspired by ideas in [4] and [2]. However, we elaborate some details omitted in [2]. In particular, the concept of levels of nodes in a Kripke model, see below, seems to be helpful. We treat terms more or less like Buss in [1], where, in the Introduction, a proof of completeness of Gentzen calculus with respect to semantics of *classical* logic is given. Our proof is based on the author's master's thesis [5] and uses also some ideas from Section 5.1 in the book [3]. Construction of universes of nodes of Kripke model and treatment of variables is inspired by [2].

2. Completeness

The formulas we consider may contain function and predicate symbols from a fixed countable language L, and variables from a fixed countable set. Formulas of the language L may contain logical connectives $\rightarrow$, &, $\vee$, $\neg$ and quantifiers $\forall$ and $\exists$. The equivalence connective $\leftrightarrow$ is not a basic symbol for us. We consider sequent calculus almost the same as in [4]. It is a multi-succedent calculus where succedents can contain more than one formula, but the rules for introducing $\to$, $\neg$ and $\forall$ to succedent do not allow side formulas. So after using any of these rules the succedent contains *exactly* one formula. The rest of the rules can have side formulas. The sequent is a pair of finite sets of formulas, not sequences in contrast to Takeuti, thus we do have the weakening rule, but there are no contractions and exchanges. We start with a finite sequent $\langle \Sigma \Rightarrow \Omega \rangle$, which is not provable in a theory T, and proceed to a semantic counter-example. We want to construct a semantic counter-example to provability of the sequent $\langle \Sigma \Rightarrow \Omega \rangle$ in T, which means a Kripke model and its node is such that, in that node, some evaluation of variables satisfies all formulas in $\Sigma \cup T$, but violates all formulas in Ω. Without loss of generality, we assume that no variable simultaneously has free and bound occurrences in $\Sigma \cup \Omega$ and T. The set T contains only sentences and can be infinite. We fix a countable set of variables Var_0 which contains all free variables from $\Sigma \cup \Omega$ and contains infinitely many other variables which are not used anywhere. Moreover the set Var_0 does not contain bound variables from $\Sigma \cup \Omega$ and T. As common in completeness proofs, we build a semantic counter-example from syntactical objects we deal with. We need infinite sequents in the construction and thus we have to extend the definition of provability to infinite sequents. Our proof does not use Henkin constants and our language L never varies. What varies, however is the set of admissible (free) variables. So we will have to be careful about which variables are allowed at which stage of the construction.

Definition 1. A sequent $\langle \Gamma \Rightarrow \Delta \rangle$ *is provable in infinite sense* if there are $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ such that Γ' and Δ' are finite and the sequent $\langle \Gamma' \Rightarrow \Delta' \rangle$ is provable in intuitionistic logic.

Note that, by this definition, the sequent $\langle \Sigma, T \Rightarrow \Omega \rangle$ is unprovable. At first we define the notion of saturated sequent. We say that t is *a term over a set* Var if all variables of t are in Var. Similarly, we say that φ is *a formula over a set* Var if all free variables of φ are in Var. We have no other bound variables than those that appear in $\Sigma \cup \Omega$ and T.

Definition 2. An (infinite) sequent $\langle \Gamma \Rightarrow \Delta \rangle$ *is saturated with respect to a set* Var if all formulas in Γ and Δ are formulas over the set Var and the following conditions are satisfied:

- if $\varphi \& \psi \in \Gamma$ then $\varphi \in \Gamma$ and $\psi \in \Gamma$,
- if $\varphi \vee \psi \in \Delta$ then $\varphi \in \Delta$ and $\psi \in \Delta$,
- if $\varphi \vee \psi \in \Gamma$ then $\varphi \in \Gamma$ or $\psi \in \Gamma$,
- if $\varphi \& \psi \in \Delta$ then $\varphi \in \Delta$ or $\psi \in \Delta$,
- if $\varphi \to \psi \in \Gamma$ then $\psi \in \Gamma$ or $\varphi \in \Delta$,
- if $\neg \psi \in \Gamma$ then $\psi \in \Delta$,
- if $\exists x \psi \in \Gamma$ then there exists a variable $y \in \mathrm{Var}$ which is not free in Γ and Δ and such that $\psi_x(y) \in \Gamma$,
- if $\exists x \psi \in \Delta$ then $\psi_x(t) \in \Delta$ for all terms t over the set Var,
- if $\forall x \psi \in \Gamma$ then $\psi_x(t) \in \Gamma$ for all terms t over the set Var.

Remark 1. Nothing is stated about $\to$, $\neg$, $\forall$ in Δ.

Remark 2. Whenever we apply this definition, the set Var will contain no variable bound in $\Gamma \cup \Delta$. Then it will be guaranteed that the term t and the variable y in the last three conditions are substitutable for x in φ.

Lemma 1. *Let a sequent* $\langle \Gamma \Rightarrow \Delta \rangle$ *(potentially infinite) be unprovable and the set* Var *contain all free variables in* $\langle \Gamma \Rightarrow \Delta \rangle$ *and no bound variable from* $\langle \Gamma \Rightarrow \Delta \rangle$*.* Var *contain infinitely many variables which do not occur in* $\Gamma \cup \Delta$*. Then there exists an unprovable sequent* $\langle \Pi \Rightarrow \Lambda \rangle$ *which is saturated with respect to the set* Var *and such that* $\Gamma \subseteq \Pi$ *and* $\Delta \subseteq \Lambda$ *and* Π *and* Λ *are created only from substitutional instances of subformulas of the formulas in* Γ *and* Δ*.*

Proof. Let $[\varphi_i, t_j]$ be an enumeration of all pairs where φ_i is a formula over the language L with all free variables over the set Var and t_j is a term over the set L with all free variables over the set Var. Likewise, we enumerate all terms over the set Var as $t_0, t_1, t_2, \ldots$. We enumerate all pairs $[\varphi_i, t_j]$ by a diagonal enumeration such that every pair is infinitely repeated. In each stage of the construction we consider one such pair. We proceed in stages and construct sets Γ_n and Δ_n. At the beginning we put $\Gamma_0 = \Gamma$, $\Delta_0 = \Delta$. In stage n we already have Γ_n and Δ_n. Treat the n-th pair $[\varphi_i, t_j]$ and construct the sets Γ_{n+1} and Δ_{n+1}. Note that, by unprovability of $\langle \Gamma \Rightarrow \Delta \rangle$, φ_i cannot be both in Γ_n and in Δ_n because of the property of unprovability of sequent $\langle \Gamma_n \Rightarrow \Delta_n \rangle$.

For $\varphi_i \notin \Gamma_n \cup \Delta_n$ we put $\Gamma_{n+1} = \Gamma_n$ and $\Delta_{n+1} = \Delta_n$.

For $\varphi_i \in \Gamma_n$ we do:

- if $\varphi_i = \psi \& \chi$ then $\Gamma_{n+1} = \Gamma_n \cup \{\psi, \chi\}$ and $\Delta_{n+1} = \Delta_n$,
- if $\varphi_i = \psi \vee \chi$ then

  $$\Gamma_{n+1} = \begin{cases} \Gamma_n \cup \{\psi\} & \text{if } \langle \Gamma_n, \psi \Rightarrow \Delta_n \rangle \text{ is unprovable} \\ \Gamma_n \cup \{\chi\} & \text{otherwise} \end{cases}$$

  and $\Delta_{n+1} = \Delta_n$. Recall, that $\Gamma_n \cup \{\psi\}$ or $\Gamma_n \cup \{\chi\}$ must preserve the property of unprovability. We know that if the sequent $\langle \Gamma_n, \psi \vee \chi \Rightarrow \Delta_n \rangle$ is unprovable, then one of the sequents $\langle \Gamma_n, \psi \Rightarrow \Delta_n \rangle$, $\langle \Gamma_n, \chi \Rightarrow \Delta_n \rangle$ will have to be unprovable from the definition,
- if $\varphi_i = \psi \rightarrow \chi$ then

  $$\begin{cases} \Gamma_{n+1} = \Gamma_n \cup \{\chi\},\ \Delta_{n+1} = \Delta_n & \text{if } \langle \Gamma_n, \chi \Rightarrow \Delta_n \rangle \text{ is unprovable} \\ \Delta_{n+1} = \Delta_n \cup \{\psi\},\ \Gamma_{n+1} = \Gamma_n & \text{otherwise} \end{cases}$$

- if $\varphi_i = \neg\psi$ then $\Delta_{n+1} = \Delta_n \cup \{\psi\}$ and $\Gamma_{n+1} = \Gamma_n$,
- if $\varphi_i = \exists x \psi$ then $\Gamma_{n+1} = \Gamma_n \cup \{\psi_x(y)\}$, where $y \in$ Var is chosen so that y is not free in Δ_n and Γ_n, and $\Delta_{n+1} = \Delta_n$,
- if $\varphi_i = \forall x \psi$ then $\Gamma_{n+1} = \Gamma_n \cup \{\psi_x(t_j)\}$ for term t_j and $\Delta_{n+1} = \Delta_n$.

For $\varphi_i \in \Delta_n$ we do:

- if $\varphi_i = \psi \& \chi$ then

  $$\Delta_{n+1} = \begin{cases} \Delta_n \cup \{\psi\} & \text{if } \langle \Gamma_n \Rightarrow \Delta_n, \psi \rangle \text{ is unprovable} \\ \Delta_n \cup \{\chi\} & \text{otherwise} \end{cases}$$

  and $\Gamma_{n+1} = \Gamma_n$,

- if $\varphi_i = \psi \vee \chi$ then $\Delta_{n+1} = \Delta_n \cup \{\psi, \chi\}$ and $\Gamma_{n+1} = \Gamma_n$,
- if $\varphi_i = \exists x \psi$ then $\Delta_{n+1} = \Delta_n \cup \{\psi_x(t_j)\}$ for term t_j and $\Gamma_{n+1} = \Gamma_n$,
- if φ_i is an implication, negation or a universally quantified formula, we put $\Gamma_{n+1} = \Gamma_n$ and $\Delta_{n+1} = \Delta_n$.

  $$\Pi = \bigcup_{n=0}^{\infty} \Gamma_n \qquad \text{and} \qquad \Lambda = \bigcup_{n=0}^{\infty} \Delta_n.$$

  The sequent $\langle \Pi \Rightarrow \Lambda \rangle$ is unprovable and saturated with respect to the set Var due to the construction.

□

Now, we construct the desired Kripke model and we show that the model has the required properties. The idea of the model is the same as in [1]. First we define sets Var_i. Recall, that the set Var_0 is an infinite countable set of variables. It contains all free variables from Γ and Δ, but does not contain bound variables from Γ, Δ and T and contains infinitely many other variables. Var_k, Var_j are fixed as infinite countable sets disjoint with Var_0 which do not contain any bound variables from Γ, Δ and T and $\forall k, j \in \mathbb{N}$, $k \neq j$ Var_j and Var_k are disjoint.

We take unprovable saturated sequents with respect to the set $\bigcup_{k=0}^{i} \mathrm{Var}_k$ on the level i. Those sequents we write $[\langle \Gamma \Rightarrow \Delta \rangle, i]$ and these pairs are worlds (nodes) in constructed Kripke model. We construct $[\langle \Pi \Rightarrow \Lambda \rangle, 0]$ an unprovable saturated sequent to the sequent $\langle \Gamma, T \Rightarrow \Delta \rangle$ according to the Lemma 1. Kripke model is not empty because we have a sequent $\langle \Pi \Rightarrow \Lambda \rangle$ on the level 0, which is saturated with respect to the set Var_0 and this sequent is over the set $\langle \Gamma, T \Rightarrow \Delta \rangle$. The model is denoted by $\mathcal{K} = \langle W, \leq, \rho \rangle$, where ρ assigns elements of W structures. The accessibility relation is defined as follows:

$$[\langle \Gamma \Rightarrow \Delta \rangle, i] \leq [\langle \Gamma_1 \Rightarrow \Delta_1 \rangle, j] \Leftrightarrow \Gamma \subseteq \Gamma_1 \& i \leq j.$$

This relation is quasi ordering. The accessible universe is always bigger or equal to the starting universe due to the condition $i \leq j$. Realization of function symbols in the structure $\mathcal{D}$ on the level i is defined: $F^{\mathcal{D}}(t_1, \ldots, t_n)$ is the term $F(t_1, \ldots, t_n)$. The universe on the level $i+1$ is potentially bigger than the universe on the level i, hence the realization which was in $\mathcal{D}$ on the level i is preserved. We define the realization of predicate symbols in the structure $\mathcal{D}$ on the level i belongs to the node $[\langle \Gamma \Rightarrow \Delta \rangle, i]$:

$$P^{\mathcal{D}} = \left\{ [t_1, \ldots, t_n];\ t_1, \ldots, t_n \text{ terms over } \textstyle\bigcup_{k=0}^{i} \mathrm{Var}_k \text{ and } P(t_1, \ldots, t_n) \in \Gamma \right\}.$$

Lemma 2. *Let s be a term, $\mathcal{D}$ be a structure on the level i and terms $t_1, \ldots, t_n$ be built from variables over the set $\bigcup_{k=0}^{i} \mathrm{Var}_k$. Then $s^{\mathcal{D}}[t_1, \ldots, t_n] = s(t_1, \ldots, t_n)$.*

Proof. The prove is the induction on the complexity of the term. Square brackets designate valuation of variables and round brackets designate substitution. If s is a variable, then $s^{\mathcal{D}}[t_1, \ldots, t_n] = (t_1, \ldots, t_n)(s) = t_j$ which is the same that we appoint t_j instead of x_j. The first equality is from Tarski's definition. If s is $F(s_1, \ldots, s_k)$, then

$$(F(s_1, \ldots, s_k))^{\mathcal{D}}[t_1, \ldots, t_n] = r(F)(s_1^{\mathcal{D}}[t_1, \ldots, t_n], \ldots, s_k^{\mathcal{D}}[t_1, \ldots, t_n]).$$

The function r is defined for the structure $\mathcal{D}$ in the L such that $r(F)$ is an n-ary operation on the set D (function from D^n to the D), hence the element $r(F)$ is realization of function F in the structure $\mathcal{D}$. We will give terms t_j to separately terms what is the same as give terms at the same time (due to the induction assumption), hence we will obtain:

$$r(F)(s_1^{\mathcal{D}}[t_1,\dots,t_n],\dots,s_k^{\mathcal{D}}[t_1,\dots,t_n]) = F\left(s_1(t_1,\dots,t_n),\dots,s_k(t_1,\dots,t_n)\right).$$

□

Lemma 3. *For all levels i, for all evaluations e on this level such that $t_1,\dots,t_n$ are values of the evaluation e of the variables $x_1,\dots,x_n$, for all formulas $\varphi\left(x_1,\dots,x_n\right)$ and for all nodes $[\langle\Gamma\Rightarrow\Delta\rangle, i]$ is valid:*

$$[\langle\Gamma\Rightarrow\Delta\rangle, i] \Vdash \varphi(x_1,\dots,x_n)[e], \text{ if } \varphi_{x_1,\dots,x_n}(t_1,\dots,t_n)\in\Gamma,$$
$$[\langle\Gamma\Rightarrow\Delta\rangle, i] \nVdash \varphi(x_1,\dots,x_n)[e], \text{ if } \varphi_{x_1,\dots,x_n}(t_1,\dots,t_n)\in\Delta.$$

Proof. The prove is the induction on the complexity of the formula φ. Let φ be an atom $P(s_1(\underline{x}),\dots,s_k(\underline{x}))$ where $\underline{x}$ means n-tuples. Let α be a node of Kripke model $[\langle\Gamma\Rightarrow\Delta\rangle, i]$. Then the following holds:

$$\alpha \Vdash P(s_1,\dots,s_k)[e] \Leftrightarrow P(s_1(\underline{x}),\dots,s_k(\underline{x}))(t_1,\dots,t_n)\in\Gamma,$$

which is equivalent to $P(s_1(\underline{t}),\dots,s_k(\underline{t}))\in\Gamma$. We obtain the last equivalence due to the Lemma 2. We have $\left[s_1^{\mathcal{D}}[e],\dots,s_k^{\mathcal{D}}[e]\right]\in P^{\mathcal{D}}$, hence we have $[s_1(\underline{t}),\dots,s_k(\underline{t})]\in P^{\mathcal{D}}$, thus according to the definition of the realization of predicate symbols in the structure $\mathcal{D}$ we have for the atomic formula φ $\alpha\Vdash\varphi[e]\Leftrightarrow\varphi\in\Gamma$.

Let $\varphi=\psi\&\chi$ and $\varphi\in\Gamma$. We have $\psi\in\Gamma$ and $\chi\in\Gamma$ due to saturatedness, hence we obtain $[\langle\Gamma\Rightarrow\Delta\rangle, i]\Vdash\psi$ and $[\langle\Gamma\Rightarrow\Delta\rangle, i]\Vdash\chi$ by the induction assumption, so $[\langle\Gamma\Rightarrow\Delta\rangle, i]\Vdash\varphi$.

Let $\varphi\in\Delta$. We have to show φ is not valid in the corresponding nodes. We have $\psi\in\Delta$ or $\chi\in\Delta$ due to saturatedness of the sequent. According to the definition of saturated sequent, the conjunct which is not valid is in the set Δ, hence we have $[\langle\Gamma\Rightarrow\Delta\rangle, i]\nVdash\varphi$ by the induction assumption.

Let $\varphi=\psi\vee\chi$ and $\varphi\in\Gamma$. We have $\psi\in\Gamma$ or $\chi\in\Gamma$ due to saturatedness of Γ. We have $[\langle\Gamma\Rightarrow\Delta\rangle, i]\Vdash\psi$ or $[\langle\Gamma\Rightarrow\Delta\rangle, i]\Vdash\chi$ by the induction assumption, so $[\langle\Gamma\Rightarrow\Delta\rangle, i]\Vdash\varphi$.

Let $\varphi\in\Delta$, hence $\psi\in\Delta$ and $\chi\in\Delta$, since we gain $[\langle\Gamma\Rightarrow\Delta\rangle, i]\nVdash\psi$ and $[\langle\Gamma\Rightarrow\Delta\rangle, i]\nVdash\chi$ by the induction assumption, hence $[\langle\Gamma\Rightarrow\Delta\rangle, i]\nVdash\varphi$.

Let $\varphi=\psi\to\chi$ and $\varphi\in\Gamma$. We can consider the sequent $[\langle\Gamma'\Rightarrow\Delta'\rangle, j]$ which is accessible from the node $[\langle\Gamma\Rightarrow\Delta\rangle, i]$. Then $\psi\to\chi\in\Gamma'$ because $\Gamma\subseteq\Gamma'$, hence from the saturatedness $\chi\in\Gamma'$ or $\psi\in\Delta'$. Let $[\langle\Gamma'\Rightarrow\Delta'\rangle, j]\Vdash\psi$. We check that $[\langle\Gamma'\Rightarrow\Delta'\rangle, j]\Vdash\chi$. If $\chi\in\Gamma'$, then is valid $[\langle\Gamma'\Rightarrow\Delta'\rangle, j]\Vdash\psi\to\chi$ by the induction assumption. If $\psi\in\Delta'$, then the induction assumption gives that formula ψ is not valid, but we assumed, that ψ is valid, hence this case will not happen. Hence we have $[\langle\Gamma\Rightarrow\Delta\rangle, i]\Vdash\psi\to\chi$.

Let $\varphi\in\Delta$. We can consider the sequent $[\langle\Gamma,\psi\Rightarrow\chi\rangle, i]$. We expand this sequent to the saturated sequent with respect to the set $\bigcup_{k=0}^{j}\mathrm{Var}_k$ where $j=i+1$. So we obtain the sequent $[\langle\Gamma',\psi\Rightarrow\chi,\Delta'\rangle, j]$ which is accessible from the node $[\langle\Gamma\Rightarrow\Delta\rangle, i]$. Γ' contains subformulas of ψ and Δ' includes subformulas of χ. We have $[\langle\Gamma',\psi\Rightarrow\chi,\Delta'\rangle, j]\Vdash\psi$ and $[\langle\Gamma',\psi\Rightarrow\chi,\Delta'\rangle, j]\nVdash\chi$ by the induction assumption, hence there is an accessible world where the assumption of implication is valid and the conclusion is not valid, so $[\langle\Gamma\Rightarrow\Delta\rangle, i]\nVdash\varphi$.

Let $\varphi = \neg\psi$ and $\varphi \in \Gamma$. We can consider any accessible saturated sequent $[\langle \Gamma' \Rightarrow \Delta' \rangle, j]$, $\Gamma \subseteq \Gamma'$, hence $\neg\psi \in \Gamma'$. We obtain $\psi \in \Delta'$ from the saturatedness, thus by the induction assumption $[\langle \Gamma' \Rightarrow \Delta' \rangle, j] \nVdash \psi$, hence ψ is not valid in any accessible world, therefore $[\langle \Gamma \Rightarrow \Delta \rangle, i] \Vdash \varphi$.

Let $\varphi \in \Delta$. We can consider the sequent $[\langle \Gamma, \psi \Rightarrow \neg\psi \rangle, j]$ which we can complete to the saturated sequent $[\langle \Gamma', \psi \Rightarrow \neg\psi, \Delta' \rangle, j]$ and $j = i + 1$. We have $[\langle \Gamma', \psi \Rightarrow \neg\psi, \Delta' \rangle, j] \Vdash \psi$ and this sequent is accessible from the sequent $[\langle \Gamma \Rightarrow \Delta \rangle, i]$, hence there is an accessible sequent where ψ is valid, therefore $[\langle \Gamma \Rightarrow \Delta \rangle, i] \nVdash \varphi$.

Let $\varphi = \exists x\psi$ and $\varphi \in \Gamma$. The sequent $[\langle \Gamma \Rightarrow \Delta \rangle, i]$ is saturated, so we have that for some $y \in \bigcup_{k=0}^{i} \mathrm{Var}_k$, $\psi(y) \in \Gamma$. We have $[\langle \Gamma \Rightarrow \Delta \rangle, i] \Vdash \psi(y)$ by the induction assumption, so $[\langle \Gamma \Rightarrow \Delta \rangle, i] \Vdash \varphi$.

Let $\varphi \in \Delta$. The sequent $[\langle \Gamma \Rightarrow \Delta \rangle, i]$ is saturated, thus we have $[\langle \Gamma \Rightarrow \Delta \rangle, i] \nVdash \psi_x(t)$ for all terms on this level, hence $[\langle \Gamma \Rightarrow \Delta \rangle, i] \nVdash \varphi$.

Let $\varphi = \forall x\psi$ and $\varphi \in \Gamma$. We can consider any sequent $[\langle \Gamma' \Rightarrow \Delta' \rangle, j]$ which is accessible from the sequent $[\langle \Gamma \Rightarrow \Delta \rangle, i]$. We have $\forall x\psi \in \Gamma'$ because $\Gamma \subseteq \Gamma'$. We obtain $[\langle \Gamma' \Rightarrow \Delta' \rangle, j] \Vdash \psi_x(t)$ for all terms on this level due to saturatedness and the induction assumption, hence $[\langle \Gamma \Rightarrow \Delta \rangle, i] \Vdash \varphi$.

Let $\varphi \in \Delta$. We have the sequent $[\langle \Gamma \Rightarrow \Delta \rangle, i]$ and consider the sequent $\langle \Gamma \Rightarrow \psi_x(y) \rangle$. We can find to this sequent the saturated sequent on the level i+1 $[\langle \Gamma' \Rightarrow \psi_x(y), \Delta' \rangle, i+1]$. We have $[\langle \Gamma' \Rightarrow \psi_x(y), \Delta' \rangle, i+1] \nVdash \psi_x(y)$ by the induction assumption, thus we have $[\langle \Gamma \Rightarrow \Delta \rangle, i] \nVdash \varphi$. We remark that y is a new variable.

We showed that the constructed model has required properties. Any node $[\langle \Gamma \Rightarrow \Delta \rangle, i]$ in model $\mathcal{K}$ satisfies all formulas from the antecedent and does not satisfy any formula from the succedent of the sequent $\langle \Gamma, T \Rightarrow \Delta \rangle$. □

Recall, that $\langle \Sigma \Rightarrow \Omega \rangle$ is an unprovable sequent in theory T, thus according to the Lemma 1 we can extend the sequent $\langle \Sigma, T \Rightarrow \Omega \rangle$ to the saturated with respect to the appropriate levels. We obtain the saturated sequent $[\langle \Gamma \Rightarrow \Delta \rangle, i]$ such that $\Sigma \cup T \subseteq \Gamma$ and $\Omega \subseteq \Delta$. We construct the model $\mathcal{K} = \langle W, \leq, \rho \rangle$ where all formulas from the antecedent of the sequent $\langle \Sigma, T \Rightarrow \Omega \rangle$ are valid and all formulas from the succedent are not valid due to the Lemma 3. Now, we summarize the statement in the Theorem 1.

Theorem 1. *(About strong completeness of Gentzen calculus with respect to Kripke semantics)*

Let $\langle \Sigma \Rightarrow \Omega \rangle$ be a sequent which is valid in all Kripke structures for intuitionistic logic where all axioms of a theory T are valid in all nodes. Hence the sequent $\langle \Sigma \Rightarrow \Omega \rangle$ is provable by cut-free proof in the theory T in intuitionistic logic, so Gentzen calculus is complete with respect to Kripke semantics.

3. Conclusion

We have proved the completeness of Gentzen calculus for intuitionistic logic, as defined in [4] with respect to Kripke semantics. We have elaborated all details of the proof using the notion of a saturated sequent.

Note that while the definition of Kripke semantics allows an accessibility relation of any order type (and of any cardinality), our construction yields a Kripke structure whose accessibility relation is well-founded. In fact, every linear set of nodes in the structure we constructed is finite or its order type is ω. We find this fact quite interesting.

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