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WALLY AXIOMATICS OF BRANCHING CONTINUATIONS

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ABSTRACT
The motivation behind axiomatization of Branching continuations is present first. Then semantics of Branching continuations are briefly explained. Thereafter we give a short introduction to the axiomatization of temporal logics. At the end we tackle the question how to axiomatize Branching continuations and present some preliminary results and observations.
Keywords: branching space-time, branching continuations, axiom, syntax, temporal logic.

1. Introduction

Branching continuations (BCont) is a temporal logic introduced by T. Placek in his article [6]. It represents a descendant of the Branching time logics of A. Prior and its direct predecessor is the so called Branching space-time logic (BST) from N. Belnap[1]. Although BST was extensively studied since its introduction, there weren't presented any clear axioms of the theory neither in Hilbert-style, nor in Gentzen-style axioms. Every account of this logic and its relatives is at most accompanied by a definition starting with the words "A model of the theory of branching spacetime is a pair $\mathcal{W} = \langle W, \leqslant \rangle$ that satisfies the following axioms" followed by a list of conditions concerning the structure of the model. Thus some axioms are present but their syntactical form is hidden in plain sight the same way as Wally[1] usually is. This work presents a generally informal attempt to find these hidden axioms of BCont and thereby externalize the debate on this topic from the 'Prague dynamic group'[2].

[1] For those who are not familiar with Wally, it is a cartoon figure used in childrens' books. He is hidden in a crowd and one must find him, which is not always easy. Although he has a quite unique appearance, the reader is deceived by similarly dressed false bait. Please look for "Where's Wally?" for additional information and pictures.

[2] Many thanks go to O. Majer, M. Peliš and others that participated in the debates concerning the nature of BCont and BST. Also my thanks must go to T. Placek and L. Wroński for comments on the paper.

2. Branching continuations

There are two reasons why BCont is used for the first attempt to make an axiomatization of branching spatiotemporal models derived from Branching space-time. The first reason is that the language of BCont has operators that seem to be (almost) the same as those encountered in tradictional temporal logics[8]. Belnap in [2] did introduce propositional temporal language into BST. Hence one could attempt similar work with BST.

However, the second reason is that we want to stay close to the use of temporal notions in natural language. Our everyday communication seems to favour expressions and concepts closer to l-events from BCont than histories from BST. When speaking about the future or about contrafactuals, our statements contain some reference events to point to the context of our statement and then the main claim (as in: "As soon as I push this button, the doors will open." or "If I had won the lottery, I would have found true happiness."). The maximality of histories in BST[3] seems to be less in accord with our use of language.

For these reasons, although a BST attempt might be easier, we have chosen BCont as the starting point. Let us introduce the basic notions connected to BCont.

2.1 BCont structure

The fundament of branching theories is $(W, \leqslant)$ where W is a set of so called point-events and $\leqslant$ is their partial ordering.

Definition 1 (Snake-link[6][4]).
The properties and basic definitions of snake-links:

(1) $\langle e_1, e_2, \ldots, e_n, \rangle \subseteq W$ $(1 \leqslant n)$ *is a snake-link iff*

$$\forall i : 0 < i < n \rightarrow (e_i \leqslant e_{i+1} \lor e_{i+1} \leqslant e_i)$$

(2) *A snake-link is above (below)* $e \in W$ *if every element of it is strictly above (below)* e.

(3) *Let* $W' \subseteq W$ *and* $x, y \in W'$. x *and* y *are snake-linked in* W' *iff there is a snake-link* $\langle e_1, e_2, \ldots, e_n, \rangle$ *such that such that* $x = e_1$ *and* $y = e_n$ *and* $e_i \in W'$ *for every* $i : 0 < i \leq n$.

(4) *For* $x, y \in W$, x *and* y *are snake-linked above* e *(*$x \approx_e y$*), iff there is a snake-link* $\langle e_1, e_2, \ldots, e_n, \rangle$ *above* e *such that* $x = e_1$ *and* $y = e_n$.

Obviously the fourth definition is a special case of the third and can be altered for other relations. The relation $\approx_e$ is reflexive, symmetrical, and transitive, hence an equivalence relation on the set $W_e = \{e' \in W | e < e'\}$.

Definition 2 (Set of possible continuations[6]).
Set of possible continuations of e *(*Π_e*) is the partition of* W_e *induced by the relation* $\approx_e$.

$\forall e < x : \Pi_e \langle x \rangle$ *is the unique continuation of* e *to which the given* x *belongs.*

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3 The definition of a history in BST is "h is a directed subset of W, and no proper superset of h has this feature"[1]. May the reader forgive, but we do not have the space to fully introduce BST.

4 The source of all cited results is placed next to the result's name.

Definition 3 (Set CE of choice events[6]**).**
For $e \in W$, $e \in CE$ iff $card(\Pi_e) > 1$.

Definition 4 (Consistency[6]**).**
For $e, e' \in W$, let there be $W_e := \{x \in W | \forall c(c \in CE \wedge c < e \rightarrow c < x)\}$ and a similar for e'. Then e, e' are consistent *iff they are snake-linked within $W_e \cup W_{e'}$. A set $A \subseteq W$ is then consistent if every two elements of A are consistent. A set A is inconsistent iff it is not consistent.*

Definition 5 (Large events, l-events[6]**).**
$A \subseteq W$ is an l-event *iff $A \neq \emptyset$ and A is consistent.*

Definition 6 (Model of BCont[6]**).**
A model of the theory of BCont is a pair $\mathcal{W} = (W, \leqslant)$ that satisfies the following axioms:

(1) *W is a non-empty set partially ordered by $\leqslant$;*
(2) *the ordering $\leqslant$ is dense on W;*
(3) *W has no maximal elements;*
(4) *every lower bounded chain $C \subseteq W$ has an infimum;*
(5) *if a chain $C \subseteq W$ is upper bounded and $C \leq b$, then there is a unique minimum in $\{e \in W | C \leqslant e \wedge e \leqslant b\}$;*
(6) *for every $x, y, e \in W$, if $e \nleqslant x$ and $e \nleqslant y$, then x and y are snake-linked in the subset $W_{e\nleqslant} := \{e' \in W | e \nleqslant e'\}$ of W;*
(7) *if $x, y \in W$ and $W_{\leqslant xy} := \{e \in W | e \leqslant x \wedge e \leqslant y\} \neq \emptyset$, then $W_{\leqslant xy}$ has a maximal element;*
(8) *for every $x_1, x_2 \in W$, if $\forall c : c \in CE \rightarrow c \nleqslant x_i$, then x_1, x_2 are snake-linked in the subset $W_{\ngeqslant CE} := \{e \in W | \forall c \in CE e \ngeqslant c\}$ of W.*

A large events is a consistent set of events. This is close to the use of events in natural language where possible futures are described by a set of events. For example, if we think about a possible outcome of a sea battle, we can take an l-event *A* to be the set of sentences such as { the weather is nice, the general of the red army slept well, the blue army had a bad breakfast }. These sentences point out some possible futures.

As we see, these aren't the syntactical axioms we are looking for. In search of lost axioms, we need to address the following points: how do we understand the 'axioms' of BCont, and how do we represent them in the language of BCont. We will compare these structures with those given by priorean or modal formulae with hopes of finding suitable representations of our demands. In order to do so, we need to introduce also the language of BCont.

2.2 BCont language

As a short reminder of the classical language of temporal logics, let us say that it consists usually of four temporal operators added to classical propositional logic formulae. These operators are based on the modal $\Box$ and $\Diamond$ operators. The main difference being that they are not limited to one direction of the accessibility relation. Thus we get the operators **H** and **G** being the past and future equivalent of $\Box$ and **F**, **P** as temporal equivalents of the diamond. They are read in the following way:

$\mathbf{F}\varphi$	*it will be the case that* φ
$\mathbf{P}\varphi$	*it was the case that* φ
$\mathbf{G}\varphi$	*it always will be that* φ
$\mathbf{H}\varphi$	*it always was that* φ

The relation between these classical operators is similar as with their modal counterparts, e.g.: $\neg\mathbf{H}\neg\varphi \equiv \mathbf{P}\varphi$. Therefore we can add just two operators ($\mathbf{H}, \mathbf{G}$) to the language of propositional logic and treat $\mathbf{P}, \mathbf{F}$ as abbreviations. There is no simple relation between the operators going into opposite directions of time. It is also possible to read the two necessity operators as incorporating the present moment, e.g.: $\mathbf{H}\varphi$ would mean 'it always was and is now that' but we do not use this interpretation.

The BCont language presented in [6], however, consists of the classical logical language, the operator $Sett$:, the operator Now :, and the temporal operators $\mathbf{P}_x, \mathbf{F}_x$. In short, $\mathbf{P}_x\varphi$ means that x units of time in the past φ is true. Similarly for $\mathbf{F}_x$. We see that these are metric tense operators and thus their behaviour is different from that of the classical operators. The operator $Sett : \varphi$ means that for all possible continuations φ is true[5]. The formula $Now : \varphi$ means that φ holds at some event contemporary to the event of evaluation. We see that these operators slightly differ from the classical temporal operators.

Only the classical operator $\mathbf{F}\varphi$ would be equivalent to a formula that is close to an actual BCont formula, namely $\exists x\mathbf{F}_x\varphi$. We can take $Sett$: as similar to $\Box$. We define $Poss : \varphi \equiv \neg Sett : \neg\varphi$ in BCont, in other words it would be equivalent to $\Diamond$. There remains $Now : \varphi$, which does not have a simple equivalent in classical temporal logic. We can observe that this exception is not a troubling one as Now : can serve, to some extent, a little bit as a nominal known from hybrid logic. These operators show to some extent how BCont focuses on natural language phenomena.

The inverse translation between the classical language of temporal logics and BCont, however, still remains a challenge. The language of BCont does not know any operator that would be equivalent to the classical operators $\mathbf{H}, \mathbf{G}$. Even without introducing the exact definition of $\mathbf{F}_x, \mathbf{P}_x$ we could show that opposed to classical temporal operators $\mathbf{F}_x \equiv \neg\mathbf{G}_x\neg$ does not hold without some additional comments. These concerns will be addressed after we introduce the definitions of BCont semantics.

We present the definition of point fulfilling a formula from the original BCont paper. That is in the form used for the so called BT+*Instants*-like semantics of BCont. This is a simple but sufficient way of capturing the general idea of our procedure. We recommend [6] or [8] for more details about BCont semantics.

Definition 7 (Evaluation points[6])**.**
Let $\mathfrak{G} = \langle \mathcal{W}, X \rangle$ be a structure for language $\mathcal{L}$, where $\mathcal{W} = \langle W, \leq, S \rangle$. Then $\langle e, A \rangle$, written as e/A, is an evaluation point in $\mathfrak{G}$ for formulas of $\mathcal{L}$ iff $\{e\} \cup A \subseteq W$ and $A \neq \emptyset$.

Definition 8 (S-t locations[6])**.**
We say that a model $\langle W, \leq \rangle$ of BCont has spatio-temporal locations iff there is a partition S of W such that

[5] In the original article the definition of $Sett$: must be a little bit more complicated as it takes into account the metric indicators of **F** and **P**. For our purpose we can omit this point at this moment.

(1) *For each l-event A and each $s \in S$, the intersection $A \cap s$ contains at most one element;*
(2) *S respects the ordering $\leq$, that is, for all l-events A, B, and all $s_1, s_2 \in S$, if all the intersections $A \cap s_1, A \cap s_2, B \cap s_1$ and $B \cap s_2$ are nonempty, and $A \cap s_1 = A \cap s_2$, then $B \cap s_1 = B \cap s_2$;*
(3) *similarly for the strict ordering;*
(4) *if $e_1 \leq e_2 \leq e_3$, then for every l-event A such that $s(e_1) \cap A \neq \varnothing$ and $s(e_3) \cap A \neq \varnothing$, there is an l-event A' such that $A \subseteq A'$ and $s(e_2) \cap A \neq \varnothing$, where $s(e_i)$ stands for a (unique) $s \in S$ such that $e_i \in s$;*
(5) *if L is a chain of choice events in $\langle W, \leq \rangle$ upper bounded by e_0 and such that $\exists s \in \in S \forall x \in L \exists e \in W : (x < e \wedge s(e) = s)$, then $\exists e^* (e^* \in \bigcap_{x \in L} \Pi_x (e_0) = s)$.*

S is then called a set of s-t locations for $\langle W, \leq \rangle$.

Definition 9 (Ordering of s-t locations[6]).
For $s_1, s_2 \in S$, let $s_1 \lesssim s_2$ iff $\exists e_1, e_2 (e_1 \in s_1 \wedge e_2 \in s_2 \wedge e_1 \leq e_2)$.

Definition 10 (Fan of evaluation points[6]).
Let $\mathfrak{G} = \langle \mathcal{W}, X \rangle$ be a structure for $\mathcal{L}$, $\mathcal{W} = \langle W, \leq, S \rangle$, and e/A be an evaluation point in $\mathfrak{G}$ for $\mathcal{L}$.

Two l-events A_1 and A_2 of $\mathcal{W}$ are isomorphic instant-wise iff
$\forall e_1 \in A_1 \, \exists e_2 \in A_2 \; s(e_1) = s(e_2)$ *and* $\forall e_2 \in A_2 \, \exists e_1 \in A_1 \; s(e_1) = s(e_2)$.

Fan of evaluation points determined by evaluation point e/A is $e/A' \in \mathcal{F}_{e/A}$, iff e/A' is an evaluation point in $\mathfrak{G}$ and A and A' are isomorphic instant-wise.

The last thing we lack is a specific event, called the moment of use and denoted e_C, and the interval relation given for a coordinalization X(both [6]):

$$\text{int}(e_1, e_2, t) \text{ iff } X(s(e_2)) - X(s(e_1)) = t$$

We can now introduce the fulfillment clauses of BCont.

Definition 11 (Point fulfils formula [6]).
For given $e_C, e/A$ and the model $\mathfrak{M} = \langle \mathfrak{G}, \mathcal{I} \rangle$, where $\mathfrak{G}$ is a given structure, and $\mathcal{I}$ is a given interpretation, we define:
(1) *if $\psi \in$ Atoms:$\mathfrak{M}, e_C, e/A \approx \psi$ iff $e \in \mathcal{I}(\phi)$*
(2) *if ψ is $\neg\varphi : \mathfrak{M}, e_C, e/A \approx \psi$ iff it is not the case that $\mathfrak{M}, e_C, e/A \approx \varphi$;*
(3) *for $\wedge, \vee, \rightarrow$ also in the usual manner;*
(4) *if ψ is $\boldsymbol{F}_x \varphi$ for $x > 0 : \mathfrak{M}, e_C, e/A \approx \psi$ iff there are $e' \in W$ and $e^* \in A$ such that $e' \leqslant e^*$ and $int(e', e, x)$, and $\mathfrak{M}, e_C, e'/A \approx \varphi$;*
(5) *if ψ is $\boldsymbol{P}_x \varphi, x > 0 : \mathfrak{M}, e_C, e/A \approx \psi$ iff there is $e' \in W$ such that $e' \cup A \in$ l-events and $int(e', e, x)$ and $\mathfrak{M}, e_C, e'/A \approx \varphi$;*
(6) *if ψ is $Sett : \varphi : \mathfrak{M}, e_C, e/A \approx \psi$ iff for every evaluation point e/A' from fan $\mathcal{F}_{e/A}$ and $\mathfrak{M}, e_C, e/A' \approx \varphi$;*
(7) *if ψ is $Now : \varphi : \mathfrak{M}, e_C, e/A \approx \psi$ iff there is $e' \in s(e_C)$ such that $e' \cup A \in$ l-events and $\mathfrak{M}, e_C, e'/A \approx \varphi$.*

Hence we see that the negation of any of the two operators does not yield the same result as the negations of the classical **F**, **P**. To some extent this is a desired property because the original operators $\mathbf{H}, \mathbf{G}$ make reference to the whole structure while BCont semantics always work with only localized notions. A simple negation of $\mathbf{F}, \mathbf{P}$ is not satisfactory either, because then $\neg \mathbf{F}_x \neg \varphi \equiv \mathbf{G}_x \varphi$ would mean that at the point x units in the future φ holds. We will address this later by changing the operator $\mathbf{G}$ to something similar to $\mathbf{F}_x$.

We suggest therefore a different interpretation of the original definition. If $\mathbf{F}_x \varphi$ would be read as "there is a event at most x units far in the future where φ holds". This reading does not change the definition of $\mathbf{F}_x$ too much but it allows us to define $\mathbf{G}_x$ via negation and read as "there are events up to x units far in the future where φ holds". This was the only operator in need of introduction or altering, so we present the definition of altered fulfillement clauses for these operators in BT+*Instants*-like semantics of BCont.

Definition 12 (F, G, P, H in BT+*Instants*-like semantics)**.** *For given $e_C, e/A$ and the model $\mathfrak{M} = \langle \mathfrak{G}, \mathcal{I} \rangle$[6] we define:*

- *if ψ is $\boldsymbol{F}_x \varphi$ for $x > 0 : \mathfrak{M}, e_C, e/A \models \psi$ iff there are $e' \in W$, $e^* \in A$, and $x' \leq x$ such that $e' \leqslant e^*$ and $int(e', e, x')$, and $\mathfrak{M}, e_C, e'/A \models \varphi$;*
- *if ψ is $\boldsymbol{P}_x \varphi, x > 0 : \mathfrak{M}, e_C, e/A \models \psi$ iff there is $e' \in W$ and $x' \leq x$ such that $e' \cup A \in$ l-events and $int(e', e, x')$ and $\mathfrak{M}, e_C, e'/A \models \varphi$;*
- *if ψ is $\boldsymbol{G}_x \varphi$ for $x > 0 : \mathfrak{M}, e_C, e/A \models \psi$ iff for all $e' \in W$, $e^* \in A$, and $x' \leq x$ such that $e' \leqslant e^*$ and $int(e', e, x')$, $\mathfrak{M}, e_C, e'/A \models \varphi$ holds;*
- *if ψ is $\boldsymbol{H}_x \varphi, x > 0 : \mathfrak{M}, e_C, e/A \models \psi$ iff for all $e' \in W$ and $x' \leq x$ such that $e' \cup A \in$ l-events and $int(e', e, x')$, $\mathfrak{M}, e_C, e'/A \models \varphi$ holds.*

At this point we can part from BCont and see what possible tools we could use for their axiomatization.

3. Time and axioms

There are multiple accounts of axioms of temporal logic. Most of them, however, are concerned with linear temporal logic. This can be useful to some limited extent also for our purpose. We mainly use three sources for this part of the debate. Garson in his book [5] gives a clear and vivid account of modal logics also with regards to temporal notions. The second helping hand comes from a source whose name already seems promising – Axioms for branching time by Reynolds[7]. Although Reynolds focuses, as many others, on the use of temporal logics in computer science his work remains a useful source of inspiration. Third but still of golden value is the account of Burgess[4] describing the properties of temporal logics from a more philosophical perspective.

The usual focus of temporal logics treats some form of linear temporal logic for the purposes of computer science. We can borrow some ideas from these approaches but we need to focus on branching structures. For this reason we introduce also a distinction that is central to the topic of branching temporal logics. Past is always regarded as a settled case with given truth values. Valuation of formulae that speak about the future, however, presents two options. The so called Ockhamist perspective claims that we need to specify what possible course of events h we are speaking about and only then we can assign truth

6 As usual $\mathfrak{G}$ is the structure and $\mathcal{I}$ the interpretation of the model.

values. For a given event e and a course of events h, $\mathbf{F}\varphi$ is true if sometime in the future of e in h the sentence φ is true. The other option is the Peircean view that claims that we cannot assign truth values to future sentences, the exception being necessarily true sentences. The formula $\mathbf{F}\varphi$ would be read as 'for all possible h, φ is at some point true in the future'. This is a modal notion and one can make the distinction clearly visible by introducing the modal operator $\Box$ to symbolize this quantification over the set of possible h. Ockhamist logic can distinquish between the following three: for a given h, e $\mathbf{F}\varphi$, $\Box\mathbf{F}\varphi$, $\Diamond\mathbf{F}\varphi$. On the other hand it holds that $\Box\mathbf{F}_{Ockhamist}\varphi \equiv \mathbf{F}_{Peircean}\varphi$ and hence Peircean logic cannot make the same distinction. This shows how modal notions can be naturally introduced into temporal logic. We can quote the axioms of Prior's Ockhamist branching time logic (OBTL)[7]:

Definition 13 (Axioms of OBTL[7])**.**
Let p, q be propositional atoms, $\bot$ being a constant for false, φ, ψ formulae of the language of temporal logics with modal operators, then the axioms are the following.

Inference rules of substitution, modus ponens, temporal and path generalization, and an IRR-rule:

$$\frac{\varphi}{\varphi[\psi/q]} \quad \frac{\varphi, \varphi \to \psi}{\psi} \quad \frac{\varphi}{\boldsymbol{G}\varphi} \quad \frac{\varphi}{\boldsymbol{H}\varphi} \quad \frac{\varphi}{\Box\varphi} \quad \frac{(p \wedge \boldsymbol{H}\neg p) \to \varphi}{\varphi}$$

if p does not appear in φ.

The following axioms:

L0: *any propositional tautology*
L1: $\boldsymbol{G}(\varphi \to \psi) \to (\boldsymbol{G}\varphi \to \boldsymbol{G}\psi)$ *– distributivity*
L2: $\boldsymbol{G}\varphi \to \boldsymbol{GG}\varphi$ *– transitivity*
L3: $\varphi \to \boldsymbol{GP}\varphi$ *– converse*
L4: $\boldsymbol{F}\varphi \to \boldsymbol{G}(\boldsymbol{F}\varphi \vee \varphi \vee \boldsymbol{P}\varphi)$ *– branch linearity*

*and the 'mirror images' of L1 – L4 that means switching temporal operators for their duals (e.g.: **H** with **G**). Followed by modal axioms of S5:*

S1: $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ *– distributivity*
S2: $\Box\varphi \to \Box\Box\varphi$ *– transitivity*
S3: $\Box\varphi \to \varphi$ *– reflexivity*
S4: $\varphi \to \Box\Diamond\varphi$ *– "B" axiom*

and some axioms for the relation between operators:

HN1: $p \to \Box p$ *– persistence on propositional atoms*
HN2: $\Box\boldsymbol{H}\varphi \leftrightarrow \boldsymbol{H}\Box\varphi$ *– non-branching past*
HN3: $\boldsymbol{P}\Box\varphi \to \Box\boldsymbol{P}\varphi$ *–*
HN4: $\Box\boldsymbol{G}\varphi \to \boldsymbol{G}\Box\varphi$ *–*
HN5: $\boldsymbol{G}\bot \to \Box\boldsymbol{G}\bot$ *– maximality of branches*

This example is also important because of Reynolds' observation that the axioms are sound with regards to branching trees but they are not complete. He proved completeness with regards to Kamp frames and bundled trees. It is not our aim to create new semantics for BCont based on our attempt to find the appropriate axioms, hence we take this only as an indication of the property of these axioms.

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7 It is our convention to cite the origin of a given definition or theorem next to its name.

However, the main importance of this example lays in the hybrid approach to branching time. We have temporal operators mainly capturing the relations in branches (or histories from the BST vocabulary), i.e. capturing temporal relations. On the other hand we have modal operators that capture the possibilities or options, i.e. focusing not on temporal succession but on the consistency of two events. This is well expressed by Garson[5]:

> *The demand for an open future is really a demand for openness in what is determined by the present, and should not be treated as a condition on the structure of time. Those who argue for an open "future" are really interested in the structure of determination, not the structure of time.* (pg. 103–4)

This quote is worth remembering even out of our current context as it reflects also Belnap's so called "indeterminism without choice" [1], i.e. that future events are determined also by events that are space-like related to our point of evaluation. Keeping in mind that branching structures represent such studies of determinism, we can return to BCont again.

4. BCont axioms

We return to the axioms mentioned in Definition 6. Explaining some of the axioms is straightforward. BCont asks for $\leqslant$ to be a partial order. We know formulae that capture reflexivity ($[*]\varphi \to \varphi$) and transitivity ($[*]\varphi \to [*][*]\varphi$), where $*$ stands for some temporal or modal operator and the brackets indicate if it's a necessity or possibility operator.[8] However, there is no modal formula that would be able to capture antisymmetry. A glimpse of hope comes from hybrid logic as it is able to describe antisymmetric structures thanks to its use of nominals[3] with $c \to \Box(\Diamond c \to c)$. Thus our attention turns to hybrid formulae. Obviously combining only the set of future (**F**, **G**) or past (**P**, **H**) temporal operators is not sufficient as it would be equivalent to the use of modal operators and thus futile. This extends, however, even to the group as a whole. A troubling bisimilarity persists even with all six operators from Reynolds. Let us have two structures. The first is an infinite line of events, where alternatively φ or $\neg\varphi$ is true. The second has two distinct events with a symmetric accessibility relation between them and φ holds in one event and $\neg\varphi$ holds in the other. As one can easily see, these two are bisimilar. We would need something as nominals from hybrid logic in order to distinguish between those two structures and we see later that BCont's language can give us such tools and we attempt to find an axiom capturing antisymmetry later.

Thankfully formulae can quite easily capture density, the second property mentioned in the Definition 6. The formula for density is $[*][*]\varphi \to [*]\varphi$.

In order to capture that there is no maximal point we could refer to the negation of the formula $G\bot \to FG\bot$ mentioned by Venema[9]. Venema's formula claims that there is a maximal point and although it was meant for linear temporal models, we could use it to describe our "temporal" part. A similar purpose is fulfilled also by the formula $\mathbf{G}\bot \to$

[8] We need to substitute the symbol $*$ for operators from the language of BCont in order to investigate the final form of the axioms. At this moment this notation is sufficient.

$\rightarrow \Box\mathbf{G}\bot$ from Definition 13. Our preference, however, lays with $\mathbf{G}\varphi \rightarrow \mathbf{F}\varphi$ from [4]. This formula might be also laden with the burden of being meant for linear flows of time but it does not introduce a new symbol into our language.

The following two axioms, the fourth and fifth, are of a particular nature. They speak about the structure with respect to chains of events. So far we did not manage to find a way how to work with propositional formulae in order to speak about subsets of point events with specific properties nor how these chain properties could be described in the context of the whole structure.

The sixth axiom tells us that every event has at most one antinuation (or in other words there is no branching towards the past). This is captured by one of the Reynolds' formulae, namely HN2: $\Box\mathbf{H}\varphi \leftrightarrow \mathbf{H}\Box\varphi$.

Axiom number seven uses the notion of a choice point. That should be a point allowing to distinquish at least two possible and distinct futures. Although we can capture the existence of two distinct futures by sentences such as $\Diamond\mathbf{G}\varphi \wedge \Diamond\mathbf{G}\neg\varphi$. However, in this case we are speaking only about a specific type of distinct possibilities and for example eliminates the possibility of alternating valuations. We postpone further attempts until we find a formal tool capable of handling the idea of choice points.

We see that finding equivalents to the original BCont axioms encounters some significant difficulties.

At this point we can attempt to formulate the axioms in the language of BCont. The first axiom of BCont was composed of multiple demands. In the language of BCont reflexivity would be represented with $Sett: \varphi \rightarrow \varphi$. In this way we do capture the demanded reflexivity of the relation without risking reflexive temporal relations.[9]

Transitivity is already a property common to both views, hence we could use the form $[*]\varphi \rightarrow [*][*]\varphi$, where $[*] \in \{\mathbf{H}_x, \mathbf{G}_x, Sett:\}$.[10] This axiom, however, won't have the same meaning as in 13 as we use the limited reach temporal operators of BCont. While the classical operators are not limited in their reach per se, their BCont versions speak only about a limited part of the structure. We can reformulate the axiom schema for the temporal operators as $[*]_x\varphi \rightarrow [*]_y[*]_z\varphi$, where $y + z \leq x$. In other words the antecedent tells us on what scale it "guarantees" that a limited form of transitivity works, see Example 14.

Example 14. *Let $\mathbf{H}_5\varphi$. This means 5 units or less in the future φ is true. Therefore sentences like $\mathbf{H}_1\mathbf{H}_2\varphi \rightarrow \mathbf{H}_3\varphi$ or $\mathbf{H}_3\mathbf{H}_1\varphi \rightarrow \mathbf{H}_4\varphi$ are true also. We see that the antecedent not only sets an upper bound on the sum of the spans of the iterated operators.*

Earlier antisymmetry presented a great problem and we promised an attempt to solve it by using the language of BCont. However, our attempts face similar problems as were present in the previous section. We could benefit either from the nature of the BCont temporal operators or from the new operator $Now:$. These operators do not end up being

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9 Looking at it from a partly semantic viewpoint, take the Definition 11 we see that if $Sett: \varphi$ holds for an event e/A as we can have a l-event $A = e$. Therefore also φ would hold in e as every l-event A' that is isomorphic instant-wise to A is equal to e. The use of the $Sett:$ operator allows us not to rely only on temporal operators like $\mathbf{F}_x$ or $\mathbf{P}_x$ and hence does allow us to elude the risk of a closed time-like curve.

10 In the following parts $*$ fulfills the same role as earlier. If we use a subscript it merely serves to point out some limitations for operators with subscripts. For example, $[*]_x$ could be replaced with $\mathbf{H}_x$ or $\mathbf{G}_x$.

that helpful as we would hope. Our attempts fail on the fact that all the operators speak only about relations but cannot fix a world in the same way as nominals do in hybrid logic. We can see that in the example 15.

Example 15. *Let us have the sentence* $\varphi \rightarrow Sett\colon (Poss\colon \varphi \rightarrow \varphi)$ *inspired by the hybrid logic formula for antisymmetry*[11]*. This sentence, however, holds also in the model with two events* e_1, e_2*, which are connected with a symmetric accessibility relation and where* φ *holds in both of them and some* ψ *does hold in one of the events but does not hold in the other. The same counterexample applies if we add the* $Now\colon$ *operator and try to use* $Now\colon \varphi \rightarrow Sett\colon (Poss\colon \varphi \rightarrow \varphi)$ *and even if we add one more* $Now\colon \varphi \rightarrow Sett\colon (Poss\colon \varphi \rightarrow Now\colon \varphi)$.

Density is primarily a property of determination but also of time, therefore $[*][*]\varphi \rightarrow \rightarrow [*]\varphi$ could stay as a schema for this axiom with the addition of BCont operators $[*]_y[*]_x\varphi \rightarrow [*]_x\varphi$.

Example 16. *Let there be a linear model where* $\mathbf{G}_y\mathbf{G}_x\varphi \rightarrow \mathbf{G}_x\varphi$ *holds. We see that if* $\mathbf{G}_y\mathbf{G}_x\varphi$ *holds in a event, this claims that in the event itself and in* y *succeeding events* $\mathbf{G}_x\varphi$ *holds also. Similarly as before,* $\mathbf{G}_y$ *represents a kind of "guaranteed" size for the given sentence.*

The lack of maximal points was captured with the formula $\mathbf{G}\varphi \rightarrow \mathbf{F}\varphi$. Suddenly the limited scope of our operators presents a possible obstacle. We cannot use $\mathbf{G}_x\varphi \rightarrow \mathbf{F}_{x+1}\varphi$ because then we would rule out alternating values of φ. Thankfully we do not need to look beyond the guarantees of $\mathbf{G}_x$. We can use $\mathbf{G}_x\varphi \rightarrow \mathbf{F}_x\varphi$. It is fulfilled also in structures with maximal points when we choose x correctly but it fails for arbitrary x. If $\mathbf{G}_x\varphi$ encounters the end of a branch, however, this formula still holds but the succedent of the schema won't hold because there is no point e' demanded by Definition 12. Hence the structure cannot have maximal points.

Example 17. *Let there be a linear model with a maximal point* e *and let us assume that the schema holds in it and* φ *holds in* e*. For the last, i.e. maximal, point of the model we clearly see that the sentence* $\boldsymbol{G}_1\varphi \rightarrow \boldsymbol{F}_1\varphi$ *does not hold as* $\boldsymbol{F}_1\varphi$ *demands for a point that is in the future and is distinct from* e *(based on the fact that for some point* e' $int(e, e', 1)$ *should hold but our model does not have such a point for* e*).*

The operators $\mathbf{G}_x$ and $\mathbf{H}_x$ represent in BT+*Instants*-like semantics chains, hence we could come up with the idea to use them to interpret the fourth and fifth axiom. This property is limited to these semantics, thus axioms based on it would not be complete with regards to other semantics (for example BCont+generalized flow of time semantics). However, unless we start using some first-order or probably even second-order language, supremum or infimum still remain out of our reach.

The original sixth axiom translation into logic was Reynolds' axiom HN2: $\Box\mathbf{H}\varphi \leftrightarrow \leftrightarrow \mathbf{H}\Box\varphi$. Similarly as with the no-maximal points axiom, our BCont operators can guarantee us only a limited view of the structure. Nonetheless, they can still maintain the demanded property on this limited field of view. Thus we can reformulate the axiom as $Sett : \mathbf{H}_x\varphi \leftrightarrow \mathbf{H}_x Sett : \varphi$.

[11] The original formula is $@_i \Box (\Diamond i \rightarrow i)$

We left the investigation of the seventh axiom hoping to find some way how to express that there is a choice point. BCont does give us some new tools that can describe some properties important for the characterization of a choice point. For example, $Poss : \mathbf{F}_1\varphi \wedge$ $\wedge\ Poss : \mathbf{F}_1\neg\varphi$ brings us one step closer to a choice point as it shows two possible but distinct futures, but a choice point has to be maximal in the set of common past points of the two possibilities (which from the perspective of this formula means that the subscript should be limitely close to zero). Hence we can capture the existence of two distinct futures but not the fact that there is a choice point between them for similar reasons as we could not formalize the suprema and infima of two previous axioms.

The final form our axiom number eight took was $\Box\mathbf{H}\varphi \leftrightarrow \mathbf{H}\Box\varphi \rightarrow \mathbf{H}p \vee \mathbf{PH}p$. We saw already the localized version of axiom six and we can actually use a similar localization for this axiom. The final axiom being $Sett : \mathbf{H}_x\varphi \leftrightarrow \mathbf{H}_x Sett : \varphi \rightarrow \mathbf{H}_x p \vee \mathbf{P}_y\mathbf{H}_x p$.

Let us sum up the result with regards to the BCont structure from [6] mentioning all our final ideas.

Summary 18 (Hilbert-style Axioms of BCont)**.**

Axiom	*$\mathcal{L}$ of hybrid temporal logic*	*$\mathcal{L}$ BCont*
1		
Reflexivity	$[*]\varphi \rightarrow \varphi$	$Sett : \varphi \rightarrow \varphi$
Transitivity	$[*]\varphi \rightarrow [*][*]\varphi$ $\Box\varphi \rightarrow \Box\Box\varphi$	$[*]_x\varphi \rightarrow [*]_y[*]_z\varphi$ *with* $y + z \leq x$
Antisymmetry	*None found*	*None found*
2	$[*][*]\varphi \rightarrow [*]\varphi$	$\Box\Box\varphi \rightarrow \Box\varphi$ $[*]_y[*]_x\varphi \rightarrow [*]_x\varphi$
3	$\boldsymbol{G}\varphi \rightarrow \boldsymbol{F}\varphi$	$\boldsymbol{G}_x\varphi \rightarrow \boldsymbol{F}_x\varphi$
4	*None found*	*None found*
5	*None found*	*None found*
6	$\Box\boldsymbol{H}\varphi \leftrightarrow \boldsymbol{H}\Box\varphi$	$Sett : \boldsymbol{H}_x\varphi \leftrightarrow \boldsymbol{H}_x Sett : \varphi$
7	*None found*	*None found*
8	*Ax6* $\rightarrow \boldsymbol{H}p \vee \boldsymbol{PH}p$	*Ax6* $\rightarrow \boldsymbol{H}_x p \vee \boldsymbol{P}_y\boldsymbol{H}_x p$

We see that there are properties we did not manage to capture in our Hilbert-style axioms. We managed, however, to do some observations concerning the way how BCont could be axiomatized.

5. Summary

Branching continuations were always presented without any classical axiomatic system that would be based on axioms or inference rules. Although there never existed any explicit reasoning why it is done so, we have shown in this article a few reasons why it seems reasonable to use the original BCont approach. The structures demanded for Branching continuations have some properties that are difficult to transform into temporal formulae. Namely the structure demands antisymmetry of its accessibility relation, which is usually a difficult property to model using modal logic, some properties demand notions and concepts not present in the language of the logic, for example references to chains or choice events. We managed to translate some of the BCont "axioms" into temporal propositional formulae and we suggested a translation of formulae (and operators) from classical temporal logic to BCont. We also suggested a new interpretation of the operators $\mathbf{F}_x, \mathbf{P}_x$ in order to accomodate in BCont a version of the temporal box operators $\mathbf{G}, \mathbf{H}$. However, we did not arrive to a axiomatic system for BCont as we did not manage to capture some of the properties demanded of the structure via any propositional temporal formulae. It remains an open question if those demands can be formalized in temporal formulae altogether and what will be the properties of the resulting axiomatic system if there even is one. This paper also focused only on axioms and has left the question of inference rules aside for a while. It seems that BCont's Wally axioms remain still hidden and probably in a group of higher order temporal formulae. However, the trick to find them could be to use different goggles from the original BCont ones. We could for example take l-events as primitives (keeping in mind that a trivial l-event is a single point event) and start working with the theory in a similar way as with set theory. This switch of perspective could be all that is needed. As mentioned earlier, l-events seem to reflect to a great extent how people usually speak about time, where a multitude of events or circumstances make up the context of evaluation or use. Because BCont pays homage also to our natural use of temporal language and concepts, this shift seems as a natural thing to do.

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