2017 ACTA UNIVERSITATIS CAROLINAE PAG. 7–26
PHILOSOPHICA ET HISTORICA 2 / MISCELLANEA LOGICA

LOGICAL SPACE AND THE ORIGINS OF PLURALISM IN LOGIC

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ABSTRACT
The fact that there is a plurality of systems that we call logics makes it requisite to attempt an explanation and thorough evaluation of the role of logic. I exploit the analogical development towards the pluralism of geometry to show that both disciplines are about some kinds of space which they explicate and that we can choose with some freedom the tools for engaging in an enterprise of these disciplines. After revisiting the development of non-classical (i.e. non-Euclidian) geometries, I present logical expressivism, as coined by Robert Brandom, and, returning again to geometry, show that an analogous doctrine of geometrical expressivism can also provide a viable account of the nature and purpose of the discipline and the reasons for plurality of both geometries and logics.
Keywords: pluralism, expressivism, logical space, geometry, holism

Today we have already gotten well used to there being a plurality of logics and so it seems difficult to understand the approach of logicians before the twentieth century who were convinced that there cannot be more than one logic. Pluralism is a sign of the fruitful development that logic as a mathematical discipline has undergone. Every new logic has the potential to show us hitherto unknown possibilities of the mathematical methods which form the backbone of various logics. Besides being interesting in themselves, new logics help us see particular properties of the already established ones. The development that led to the plurality of logics surely had its rationality and has brought many interesting results, as well as enriched logic and many germane disciplines.

There are, therefore, good reasons to be happy about this pluralism and not to regard it as something which we should be bothered by. Yet, I think it still is a remarkable phenomenon which has to be philosophically reflected upon. Although we have gotten used to plurality, we have to admit that there is something paradoxical about it which cannot be so easily dismissed on the grounds that I just mentioned. We are used to all kinds of pluralism in various disciplines, yet plurality should have some common denominator and therewith also some limits[1]. And what should be clearly delimited if not logic? People may have various opinions on issues that they get to discuss, yet the very rules guiding the discussion should be rather firm if that discussion is supposed to be possible at all. And it is quite natural to think that logic should be about the rules of correct argumentation, of inferring conclusions from some premises that we agree on (such an opinion is

[1] Note that even Beall and Restall who largely helped make the idea of *logical pluralism* popular also call for imposing limits on it (see Beall (2006)).

https://doi.org/10.14712/24647055.2017.12

not only expressed in textbooks on critical thinking but also in textbooks on logic itself). The plurality of logics thus seems to indicate uncertainty about these rules, at least in the case that pluralism has no clear limits and we cannot say what can be regarded as logic at all. The naive suggestion would be that we are facing a dilemma. Either logic has somehow gone astray and is not investigating these rules of correct argumentation anymore or, worse, the plurality of logics reflects that no such rules exist (anymore?) and therefore rational debate is no longer possible[2].

The second possibility is preposterous, as rational discussion obviously is possible, even if it can get very complicated. Yet, the idea that logic might not be pursuing its real purpose is worthy of investigation. If the discipline describes something which turned out to be quite apt to change – as for practically any logical law we can think of a logic according to which it holds as well as of one with an opposite verdict – then it is probably failing to describe the very foundations of our cognition, as we can expect such foundations to be something quite stable and immune to change, whether it be foundations in civil engineering or in epistemology. So if logic can be changing its statements almost at almost the same rate as the most empirical sciences, it is probably instead only describing some quite parochial part of our epistemic apparatus. Yet, perhaps we should rethink what we regard as the purpose of logic. Or maybe we could try to somehow distinguish, among all the so called "logics", the real ones (or, even better, the real one) which truly describe the foundations of our rationality, from those mathematical systems which do not deserve such a name. Nevertheless, let us investigate for a moment two disciplines that underwent a development that was, to a considerable degree, similar to that of logic. Namely, geometry and arithmetic.

1. Pluralism in geometry and arithmetics: comparable to the case of logic?

For the moment, we will not try to specify with particular precision when two logics are rivals and how much they have to differ from each other in order to vindicate logical pluralism, should both of them be accepted as legitimate logics. There are many ways in which logics can differ. Susan Haack (see Haack (1978)) distinguished between cases when logics are indeed *rivals* (or *deviant*) with respect to one another, and the case when one of them is merely an extension of the other one. For example, intuitionistic logic is a rival of classical logic, as it shares the same language and yet pronounces different verdicts as to the validity of logical laws. On the other hand, normal modal logics (or second order logic) are mere extensions of classical logic, i.e. in the restricted language the same laws hold. I want to understand two logics as being different irrespective of Haacks's distinction between rivality and extension. The dispute about whether the law of the excluded middle holds universally is for me on the same footing as that of whether the modalities belong within the purview of logic. Intuitionistic logic, classical logic and the modal logic S5 are simply different logics in my understanding. I will purposely leave the problem of individuation of a logic and therewith that of logical pluralism in this

[2] Some think that anthropological research of inference rules in exotic cultures proves as much, see Triplett (1988) for a short discussion thereof.

rather naive and undeveloped shape, as I want it to be specified and elaborated later in the course of our investigation of the role of logic.

However we interpret it, the plurality of logics is a relatively recent phenomenon. That there is only one logic was taken for granted to the point of it not being worth mention[3]. Kant attempted to show that there are good reasons for logic to be the way it was. In fact, his systematization of logical judgments presented in the *Urteilstafel* serves as one of the foundations of his transcendental philosophy. Thus, logic in its specific form is necessary for the whole complex building of our rationality. Of course, when you start giving reasons for something that is taken for granted, you, even if only in a long run, encourage the attempts to cast doubts on these reasons and consequently also the platitudes that they were supposed to substantiate. Thus, we will soon see that Kant's arguments for why logic was the way it was also opened the possibility of countering them and eventually awake the suspicion whether logic cannot, in fact, be changed. But first let us digress a little bit from logic to geometry and arithmetic, as both Kant's position and its subsequent development display many illustrative analogies to the case of logic.

According to Kant human cognition has two sources, sensibility and understanding[4], and logic describes the fundamental features of understanding, i.e. those features without which there would be no understanding at all. Analogously, in many ways , he sees geometry and arithmetic as fundamental for our sensibility (the study of which he calls *aesthetics*), which is our ability to receive the raw materials that our understanding can work with. Let us consider to which degree we can speak of pluralism in these other allegedly fundamental disciplines and where it can stem from. We hope to achieve a better understanding of logical pluralism by considering the possible analogies between the development logic, arithmetic and geometry underwent. It is very useful to make Kant our point of departure, as he can be seen as an advocate of monism in all of these disciplines. Pluralism therefore had (and despite all the progress still has) to develop in opposition to his views.

1.1 Pluralism in arithmetics?

As is well known our inuition has, according to Kant, two basic forms which enable us to perceive any object at all – namely, space and time. And time, characterized by its one-dimensionality, is described by arithmetic, the science of number. Arithmetic shows us the necessary features of our perception of time. Despite all the controversy which this Kantian doctrine of arithmetic caused, we have to say that it does not help us to any illuminative analogy with the problem of logical pluralism because arithmetic has developed differently from logic since Kant's time and the controversy was not about the possibility

[3] This does not mean that there were historically no disputes about logic. For example, the Stoics developed a logic which differed from Aristotle's. But while on the one hand, Aristotelian logic was historically dominant, on the other hand logical pluralism as the thesis that there are more legitimate logics was still hardly seriously considered, as even the Stoics though that there is only one logic which Aristotle, unlike them, failed to describe successfully.

[4] Nur so viel scheint zur Einleitung, oder Vorerinnerung, nötig zu sein, daß es zwei Stämme der menschlichen Erkenntnis gebe, die vielleicht aus einer gemeinschaftlichen, aber uns unbekannten Wurzel entspringen, nämlich Sinnlichkeit und Verstand, durch deren ersteren uns Gegenstände gegeben, durch den zweiten aber gedacht werden (Kant (1954), A16/B30).

of developing a different arithmetic. Kant's view provoked attempts to show that this discipline has a significantly different status (most prominently Frege's logicist project) than he thought, but it did not lead to pluralism. Nobody considered the possibility of there being more possible arithmetics[5].

Clearly the connection of arithmetic with time is controversial and the idea of the linear shape of time is something which was attacked by both physicists and philosophers[6]. Furthermore, the classification of arithmetic knowledge as synthetic a priori was attacked by Frege, who launched his logicist programme in an attempt to refute it. To what degree such a programme can succeed is still disputed and these disputes are actually closely related to the problem of logical pluralism, because if we try to verify or refute the logicist thesis that arithmetic can be derived from logic, it is, of course, important to know what logic is (or, put otherwise, which logic it is that are we trying to show contains arithmetics). Some authors (notably Ian Hacking (see Hacking (1979)) or Gila Sher (see Sher (1991)) have tried to legitimize their preferred logics as leading to their preferred verdicts regarding the viability of logicism.

And yet, as we said, there is no direct analogy to logical pluralism in the form of arithmetical pluralism. There are more axiomatisations of arithmetic, such as the Peano, Robinson and Presburger axioms, yet these cannot be said to describe different arithmetics[7]. These sets of axioms are distinguished mainly because they serve as different tools for investigation of the logical properties of arithmetic, differing in two basic features. On the one hand, they can have neat formal properties which make them easier to treat (as the Presburger or Robinson axiomatic), on the other hand, they can come closer to describing the actual practice of arithmetic as Peano arithmetic or, even if we allow for a shift in logic, the second order Peano arithmetic which describes the structure of natural numbers which underlies our everyday aptitude to count and in general deal with numbers l categorically.

One more aspect of the modern treatment of arithmetic may resemble pluralism, namely the fact that, for example, the first-order Peano arithmetic has got many non-isomorphic models, i.e. it is not categorical. Do these models perhaps show us that there are different kinds of arithmetic, different ways of understanding numbers and their properties and interactions? Well, hardly. The non-standard models of Peano arithmetic (and the even more non-standard ones of Robinson arithmetic) are surely an interesting and fascinating object of mathematical study, yet they can hardly be said to constitute alternative arithmetic which could, under some circumstances, supersede the arithmetic based on the daily practice of counting and related activities. Again, just as with the plurality of arithmetic theories, the plurality of models of a given theory primarily illuminate the epistemological status of arithmetic, in particular the acceptability of logicism or formalism. The difference between the plurality of theories and that of models, speaking

5 This is not to say that there were no controversies during the history of the discipline, e.g. about the properties of zero, whether it is a number at all, etc. Yet no arithmetic pluralism has ever really arisen.

6 Among them we can mention Einstein who showed that a different idea of time is possible and even preferable to the simplistic one. Besides this, temporal logicists model time as branching.

7 Unless we see the matters from a radically formalistic point of view which would claim that every set of axioms constitutes a specific object of study (with equivalent sets of axioms constituting the same object, of course).

somewhat figuratively, is that while we tend to actively create the theories, the models are rather found by us in course of examining the properties of these theories.

Though Kant's views of arithmetic are controversial and may be legitimately questioned, we cannot say that the controversies that ensued led to arithmetic pluralism, analogous to logical pluralism. The story, though, gets more intricate with respect to geometry, as this discipline in fact developed – to a great degree due to the discussions initiated by Kant – towards a pluralism resembling the one in logic.

1.2 Pluralism in geometry

The development in geometry that led to the problematization of Kant's view of it and opened the possibility of pluralism preceded a similar process that logic was to undergo. I offer an overview of the revolution which the discipline underwent since the advent of non-euclidian geometries in Arazim (2013). For our purposes a short summary will suffice. The gist of the story is, as many have probably already guessed, that Kant thought that there is only one possible geometry while the subsequent development of the discipline proved otherwise.

Just as is the case with time, space is, according to Kant, a fundamental form of our sensibility, of our intuition, i.e. the ability to be given objects of cognition which then serve as the material to which we apply our cognitive capacities. Now, just as arithmetic describes the structure of time, so does geometry describe the structure of space. It is fundamental that space, the same as time, is not seen by Kant as an entity independent of us as, e.g. a table can be, but rather as something we create at least in the sense that it could not exist without us, something which we have to constitute to make perception possible. Yet, despite this active element in regards to space, it still seems that we do not have any freedom as to how we construe it. There is just one shape it can take and this is revealed to us thanks to our pure intuition. Euclidian geometry provides a rigorous exposition of this structure. This discipline is possible because we are endowed with the ability to have intuitions and hence to operate within the forms of sense (i.e. time and space).

Yet there was an old controversy about the fifth postulate of Euclidian geometry. One of several equivalent formulations of it is that, given a line and a point outside of it on a plane, there can be only one line on the same plane going through that point and never crossing the original one (which we then call its parallel). In the course of the nineteenth century, it was shown that you can create both a theory in which an infinite number of such lines can be drawn as well as a theory in which no such line at all can be drawn. Both subsequently earned the title of geometry[8], the first one became known as *hyperbolic geometry* and the second one as *elliptic geometry*. It turned out that these theories were not only consistent, but that they could also be reasonably applied to space (or more exactly – to our discourse about space and experience thereof, as we are acquainted with it independently of the adoption of one of the specific theories of geometry).

Elliptic geometry can be seen as a theory which describes the behaviour of lines on a surface of a sphere (a plane with 'positive' curvature), while hyperbolic geometry deals

[8] For an explanation of how this became possible, I refer the reader again to my article Arazim (2013).

with lines on the surface of a kind of valley (plane with 'negative' curvature). It is natural to feel that Euclidian geometry remains somewhat more fundamental than the others and that these other geometries are based on stretching the notion of line in an illegitimate manner. Yet, from the point of view of those geometries, it is Euclidian geometry that does not deal with real lines, but with the geodesics on a curved surface[9].

The important point for us is that in geometry we can definitely speak of pluralism. This pluralism is not as broad as the logical version (we have presented only three axiomatisations which can be legitimately used for explication of space and therefore called geometries) because there are not as many geometries as logics, yet it is also present in geometry. And it is just as troubling, particularly for someone who accepted the Kantian view regarding the character of geometrical knowledge. How can the discipline be apodictically certain if there are more possible answers to some of its fundamental questions? Obviously the emergence of pluralism forced theoreticians to seek a deeper philosophical explication of the nature and purpose of the discipline. And that is exactly what I claim to be necessary for logic, as well. At this point it will be useful to review a few of the approaches to the plurality in geometry which suggest themselves and which in fact were adopted by theoreticians of various backgrounds.

1.3 How to react?

The emergence of alternative geometries was certainly shocking. No doubt for someone who is getting to understand the development of the new geometries it at first glance also appears very surprising, as it strikes one as quite strange that such a fundamental concept as, for instance, the line can be understood in radically different ways. One of the natural reactions to a shocking event is denial, to try and convince oneself and others that it did not really happen, to try and explain it away. Indeed, many Neo-kantians in the nineteenth century tried to do exactly this, as is well recounted in chapter 3 of Coffa (1993).

In this case one, can understand what motivated many theoreticians to adopt this conservative stance. We have already mentioned that alternative geometries seem to be based mainly on stretching the notion of line. In fact, the situation reminds one of Quine's critique of those who attempt to change logic; namely, that they merely succeed in changing the subject (see Quine (1986), p. 81). After all, Beltrami, who has shown how the lines of non-classical geometries can be understood as Euclidian geodesic lines on a curved plane, intended himself to show that the only legitimate geometry is Euclidian, as the others do not really speak of lines in the proper sense of the word.

This traditionalist approach may still have its appeal. We also have to admit that, at least psychologically, Euclidian geometry probably has to be the first geometry one learns to work with. The understanding of what a line is and subsequently all the other notions, such as triangle, circle, etc., is acquired in a much simpler manner if the first geometry one learns is Euclidian. Quite similarly, note that some logics are clearly more basic from a similar psychological perspective. Many logics have an intuitive appeal, perhaps that of

9 Compare this with the difference of perspective of someone who understands the operation of addition in a standard manner and somebody who understands as a non-standard arithmetician of the kind described in Kripke (1982).

the syllogistic is particularly high. Yet, putting the syllogistic aside and considering the modern logics that came after it, it has to be said that, analogously to Euclidian geometry, classical logic is, by far, learned most easily by somebody who is getting acquainted with modern logic in general. It is no accident that the other logics – intuitionistic, the modal ones, etc. – are typically learned subsequently, in more advanced courses, as variations on and expansions of the theme of classical logic, which in many ways behaves in a much more orderly fashion and is easier to work with overall[10].

Yet, this psychological privilege of Euclidian geometry and classical logic, as important as it is for teaching either geometry or logic, could perhaps be circumvented by trying new pedagogical approaches to the alternative systems. Be that as it may, psychological facts are not truly important and fundamental for the philosophy of either logic or geometry. The question is instead whether the Euclidian geometry is prior to the other ones in some transcendental sense, i.e. whether the very notions of the non-classical geometries can be understood only through the perspective of the Euclidian one. And no argument in favour of such a thesis is at hand. Indeed, when thanks to the works of Eugenio Beltrami and Hermann von Helmholtz, the alternative systems were shown to be geometries in the sense of being capable of describing space, there seems to be no rules for somehow a priori privileging one of them, for seeing it as being somewhat more basic. Indeed, Helmholtz argued exactly that what seems to be the correct lines, the Euclidian ones, is seen as curved by those who live in, say, a world with a space not obeying Euclidian rules[11]. Here, we come to another approach to the plurality of geometries.

1.3.1 Empirisation of geometry

The first approach to the plurality of geometries consisted in strongly preferring the already established system to the emerging ones. The approach which we are going to consider now differs from the conservative one in that it does not privilege Euclidian geometry. On the other hand, it wants to get rid of the plurality just as the conservatives wanted. Hermann von Helmholtz, a German theoretician active in the second half of the nineteenth century, tends to switch between empiricist and holistic views, or at least varies the emphasis in his writings on geometry. As we already mentioned, we can find passages where he focuses on the relativity of the fundamental notions. By thought experiments he makes the reader see how a world can be imagined in which what seems to be perfect lines obeys either the elliptic, or the hyperbolic laws (or the Euclidian ones, of course). Elsewhere, however, Helmhotz seems to suggest that geometry should be regarded from now on as empiricial science[12].

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10 It is, for example typically much simpler to find out whether a given formula is a tautology by the means of the truth-tables than by the means afforded for intuitionistic logic.

11 For an exposition of the important arguments and thought experiments due to Belmtrami and Helmholtz, see again the third chapter of Coffa (1993) or their original writings, namely Beltrami (1868) and Helmholtz (1870).

12 Nehmen wir aber zu den geometrischen Axiomen noch Sätze hinzu, die sich auf die mechanischen Eigenschaften der Naturkörper beziehen … dann erhält ein solches System von Sätzen einen wirklichen Inhalt, der durch Erfahrung bestätigt oder widerlegt werden, eben deshalb aber auch durch Erfahrung gewonnen werden kann (Helmholtz (1870), p. 25).

The empiricist proposal is remarkable. When Kant asserted that geometry is synthetic a priori, the development of non-contradictory alternative systems seemed to force us rather to abandon the *synthetic* part of Kant's classification of geometry as *synthetic a priori*. We will get back to this attempt later on. Helmholtz was nevertheless pioneering the refutation of categorizing geometry as *a priori*. He insisted that the acceptance of geometrical theories should be based on empirical findings. The truth of the fifth postulate can be, according to this approach, tested, e.g. by testing the equivalent thesis that the sum of angles in a triangle equals two right angles. Thus, we can focus on measuring the angles of triangles as precisely as we can in order to establish which of the geometries indeed describes reality the way it is.

This position opens new approaches of seeing how we form our theories, yet it is also very problematic. Let us begin by pointing out its shortcomings. Testing a thesis by empirical measurement seems like a relatively straightforward procedure. Yet, as was shown in Kuhn (1962) and Quine (1951), testing even relatively parochial hypotheses, i.e. close to the margins[13] of the Quinean web, can be a relatively complicated process. What appears to be recalcitrant empirical data can be explained away in many cases. One can be stubborn enough to declare such experience a mere hallucination. The process of empirical testing is, of course, much more complicated for theses which belong more to the center of the web. Quine himself, to be sure, adds that as anything can be saved, anything can be sacrificed as well[14]. Yet at a certain point, when we endeavour to empirically test some of the really core beliefs and principles, the empirical testing becomes unintelligible rather than just very complicated because some of the core beliefs can hardly be treated as something which it makes sense to question, as they enable the enterprise of testing or even of pursuing truth in the first place.

In the case of geometry it is very implausible that we should ever consider, say, any empirical finding concerning the sum of the angles in a triangle more trustworthy than one or another geometrical theory we adhere to. To be able to perform this measurement, or a similar one, we surely have to be equipped with some tools, such as a ruler. This ruler has to convince us that what we are given is indeed a triangle. Yet, to ascertain that, for instance, the sides of this triangle are indeed not curved, we first have to verify that the ruler is itself not a little bit curved. Thus, we get into an infinite regress[15]. The idea of deciding between the geometries by empirical measurements is, therefore, ill-founded. No further development of the tools at our disposal can bring about a change of this simple fact.

Does this all mean that the idea of empirical findings playing a role in our acceptance of geometry is just a mistake committed by Helmhotz and other authors? As was already mentioned, in *Two dogmas* Quine asserts that anything can be refuted in the light of empirical findings. Helmholtz has, I believe, pointed in the right direction, but we have

[13] By *margins*, I mean, as can be expected, those parts of the web we are most prone to adjust and thus consider as very empirical.

[14] He even explicitly mentions logic as something we can review in the light of empirical data. He would hardly think otherwise about geometry in this respect.

[15] This is an idea coming from Henri Poincaré. A nice exposition of how he developed his opinions can be found in Shapiro (1996).

to examine more closely what it means to refute something on the basis of empirical findings.

1.4 Holistic approach to geometry

In order to empirically test a hypothesis one needs a lot of preliminaries. First of all, it is necessary to understand properly what the hypothesis asserts, i.e. to know its meaning. An important part thereof is to understand what would count as a refutation and what as a confirmation of the hypothesis. This requires that we have a solid theoretical background on which we can base our experiments. Only with such a background does it make sense to say that a given proposition was shown to be true or otherwise. These are not particularly surprising facts, yet oftentimes people fail to realize them.

We have just sketched an argument in the section above for the thesis that we in fact cannot, based on empirical data, really construct the necessary framework to be able to assert that one of the geometries is right or wrong. Put otherwise, we cannot reasonably ask, whether *real space* is Euclidian or otherwise. We thus cannot move from the Kantian *synthetic a priori* to *synthetic a posteriori*. Would it perhaps be possible to move instead towards *analytic a priori*? That could mean lots of different things, as the Kantian notion of analyticity, as is well documented in Coffa (1993), allows for multiple interpretations. If we say that geometry should depend on the meanings of words, we have to be careful. The meanings of the fundamental geometrical terms are not clear enough to help us judge which of the geometries is right. Should one perhaps feel that the Euclidian geometry expresses the notion of a triangle most correctly, then it would not be clear why this feeling should be taken as more than just a matter of personal preferences and idiosyncracies. In fact, the development of non-Euclidian geometries showed exactly that what is objectively determined in the geometrical concepts in our standard everyday use of them is just that which leaves the dispute between geometries undecided at the analytical level.

As was shown, e.g. by Helmholtz, if one of the geometries is legitimate, then so are the others because all of them describe space though each is based on a different understanding what space is (that is, a space with either positive, negative or neutral curvature). It is not the case that one can say that one of the geometries is true while the others are false. In fact, each can interact with physics, yet each demands physics to adapt to it - Euclidian geometry, for instance, has to be paired with a different mechanics than the elliptic one (for a more detailed description, see again Arazim (2013)). Thus, any of the geometries can be used as a valuable tool for creating broader theories that help us to understand the world better. More should be said about how these geometries specifically do this. When we achieve such an explanation, we will be in a position to understand what geometry is and what purpose it serves (contrasted with that of, e.g. mechanics) and to have an enlightening account which not only is not in conflict with the plurality, but shows why it arises. Let us try to arrive at that with the help of getting back to investigating pluralism in logic.

2. Accounting for the plurality of logics

Having reviewed pluralism in geometry we will now try to use what we have learned to help us towards an account of the plurality of logics. Eventually we hope to move towards an overall picture which will advance our understanding of both logic and geometry more than what we achieved in the previous section. Thus, we travel on the path of analogy from geometry to logic and back, always achieving some progress in understanding the respective discipline.

We began by noting that the plurality of logics can be seen as something quite disturbing (despite the mathematical benefits which, as I emphasized at the very beginning, it brings). Somebody not familiar with the modern plurality would probably expect logic, even more than geometry, to be a body of truths we cannot doubt without losing any certainty we could have and ultimately even blurring the epistemic notions such as that of doubt, certainty, knowledge, etc. We reminded oureselves of Kant as a philosopher who believed that logic (or what was considered to be logic in his time) does not by any means have its form by accident, indeed that no change thereof is possible.

Here we are clearly not just speaking about a perspective someone could have had before the rise of the many modern logics and thus not about a view which is only of note in regards to the history of the discipline. Attempts at monism were undertaken much more recently. The debate between intuitionistic and classical logic was led to great degree by monists – especially the intuitionists were convinced that they are presenting the correct logic. Later, however, the two logics got along better and have coexisted relatively peacefully for quite some time now[16], though Michael Dummett still insisted that intuitionism was not compatible with classical logic, particularly not with acceptance based on the thesis that these logics speak about something else, a position adopted by Quine and which we will discuss presently. Intuitionism, as Dummett states, consists precisely in seeing classical reasoning as illegitimate[17].

An advocacy for monism was provided by Quine (see Quine (1986)), who used a modification of his own *gavagai* argument for the purpose of refuting the possibility of logical pluralism. While that original argument (used by him on various occasions, among others in Quine (1960)) purported to show how much has to remain undetermined in translation, the variant thereof regarding logic essentially does the contrary. According to Quine, logic is a body of truths so basic that we have to impose them on somebody we are translating. As a consequence of the celebrated *principle of charity*, we have to reject a translation which renders someone as disagreeing with us about the laws of logic because it violates the maxim of translation to *save the obvious*. In spite of all the leeway that we have while translating, the logical laws and truths are unshakable for Quine, as he urges

[16] About the coexistence see, e.g. Dubucs (2008), p. 50: "... times where controversy was raging are disappearing from collective memory."

[17] "As Kreisel has emphasized, the intuitionistic philosophy of mathematics comprises two theses: a positive one and a negative one. The positive one is to the effect that the intuitionistic way of construing mathematical notions and logical operations is a coherent and legitimate one, that intuitionistic mathematics forms an intelligible body of theory. The negative thesis is to the effect that the classical way of construing mathematical notions and logical operations is incoherent and illegitimate, that classical mathematics, while containing, in distorted form, much of value, is, nevertheless, as it stands unintelligible." (Dummett (1977), p. 250)

that *Logic is built into translation more fully than other systematic departments of science* (Quine (1986), p. 82). A difference in logic cannot, according to Quine, be really stated or communicated and is, therefore, to be seen as an illusion. A logician who endeavours to devise an alternative logic only changes the subject. When someone tries to, e.g. deny that in all cases everything follows from the conjunction of a statement and its negation, we have no reason to see him as really speaking about conjunction and negation[18]. Quine sums up his position in the famous dictum about the deviant logician who, *when [he] tries to deny the doctrine, he merely changes the subject* (Quine (1986), p. 81).

By accepting Quine's viewpoint, one gets to see logic as a prison we cannot escape. We simply cannot help using the logic that we in fact use. The Quinean arguments were criticized strongly by various authors, e.g. by Dummett. I think it is a pity that Quine did not speak much about whether different people or perhaps different cultures can in fact adhere to different logics and how the people who in some (not entirely clear but, for Quine, necessary) sense *have* a different logic can communicate about it. Yet, though Quine does not speak at much length about this, we can understand from his translation argument that they could never communicate their differences. Quine models the situation after that of radical translation, which is probably the source of the problem that we are not sure whether people can adhere to different logics and whether they could somehow explain their logical idiosyncrasies to each other. The route from the perspective of radical translation to that of disputes about various logics is probably not so straightforward. One has the feeling that, should the dialogue run the way Quine envisages it, something would be left uncommunicated[19]. *Prima facie* there is a real dispute between the adherents of different logics and this seems to show that Quine does not apply his maxim *save the obvious* correctly in the case of disputes between adherents of different logics. Apparently, the laws that the logicians are in dispute about are not obvious enough, not in the sense to be something we cannot abandon if we do not want to cease speaking intelligibly and understanding the others.

It nevertheless remains unclear what exactly Quine wants to say. Perhaps his message is that we all abide by classical logic (which he always defended strongly), even though we might be confused enough to think otherwise. Or maybe he admits that we can in fact adhere to different logic, yet these differences among us cannot be effectively stated and thus can be explained (away) as a sort psychological idiosyncrasy an individual or community might have. It should be fairly obvious that Quine's presentation gives a very uncharitable picture of the discussion between the adherents to different logics. Despite the intricacies involved, it seems clear that both the intuitionistic as well as the classical logician both speak of the concepts of disjunction and negation when they are in a dispute about the validity of the principle of the excluded middle (I will try to explain later what these underlying concepts consist of). Lots of authors have therefore accused Quine of contradicting his own positions from Quine (1951). Wasn't a vital part of his holism an attack on the synthetic/analytic distinction to show that every dispute can, in a way, be

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18 Quine (1986), p. 81: "They think they are talking about negation, $\neg$, *not*; but surely the notation ceased to be recognizable as negation when they attempted at recognizing some conjunctions of the form $p.\neg p$ as true and stopped regarding such sentences as implying all others."

19 Quine would most likely think of this residuum as something idle like the Wittgensteinian beetle (see the paragraph 293 in Wittgenstein (2001)).

seen as a dispute about meanings as well as a dispute about matters of fact? Should the involvement of meaning thus be a reason for denying a debate its relevance (because the debate is then *merely about meanings and not about matters of fact*), then not just the debates about logical laws, but basically any rivalry between different scientific theories could be shown to be in this sense *idle*, which is surely an unacceptable position.

2.1 Choosing the best

I took Quine as an important example of an author arguing for the impossibility and unintelligibility of logical pluralism. As his argument is challenging and interesting, it also represents a defense of logical monism at what might be its best. I thus hope that we have at least sketched some arguments in favor of the thesis that there can indeed be more logics with different verdicts regarding the validity of some logical laws by showing the flaws of his reasoning. Yet opening the possibility of there being more correct logics clearly does not imply that logic is fundamentally arbitrary and the shape of, e.g. classical logic is as good as that of any systems of rules regarding logical constants that one might contrive. As in the case of geometry, there are very good reasons why the standard logic is the way it is and why it is accepted as such. To deny this means to subscribe to some sort of radical formalism, i.e. to the thesis that we have no understanding of logic independent of the specific mathematical theories (as say the axiomatization of classical logic) and that these theories by the same token do not relate to anything independent of them. Indeed, we can see such a tendency in Hilbert's responses to Frege in their correspondence about the status of geometry and possibilities of pluralism.[20] To suggest that anything, provided it is a consistent theory, can work as geometry (or, for that matter, logic) makes the very notion of geometry unintelligible. Yet, as Jaroslav Peregrin remarks (see Peregrin (2000)), Hilbert, as well as Frege, suffered rather from tendencies to overemphasize some relevant aspects of pluralism rather than from having incompatible, extreme positions.

While not surrendering to this kind of formalism, we can thus go on talking about a more moderate and reasonable version of logical pluralism as about a meaningful position and the rivalry of logics as a source of a reasonable kind of dispute. Thus obviously someone can adhere to classical logic while another logician can adhere to intuitionistic logic and so on. Nevertheless, as we said, logic belongs to the very fundamental layers of our conceptual schemes. Radical opposition to, say, classical logic (in the sense of asserting that completely different logical laws are valid than classical ones, i.e. not that just one or a few of the classical ones are problematic as, for example, the inuitionists claimed) can hardly be reasonably defended[21]. History shows us how harsh the classical/intuitionist dispute was, especially between Hilbert and Brouwer[22]. How have the logicians arrived at the peaceful coexistence we witness today? Have they given up the difficult debates about the nature of logic and its fundamental principles (as Dummett claims)?

[20] For a helpful overview of their debate, see Shapiro (1996)

[21] Yet, of course, the dispute between adherents of different logics can very much resemble what Wittgenstein describes in paragraph 611 of Wittgenstein (1984): "Wo sich wirklich zwei Prinzipien treffen, die sich nicht miteinander aussöhnen, da erklärt jeder den Andern für einen Narren und Ketzer."

[22] A nice oveview of this fight which involved not just arguments but also the use of power and coertion, can be found in Zach (2006).

Let us reflect on what is problematic in Quine's position once again. We can thereby hope to understand what makes us doubt the possibility of plurality among logicians. To begin with, Quine believes everyone is closed in one's own logic, i.e. that according to which one judges. Logic thus becomes, in some sense, a part of our nature which can be spelled out in two ways. The first will be discussed presently, while the other will come out as a result of our reflections at the end of the paper. The first and less fortunate was the one Quine inclined to; namely, the more literal. That is, logic is a part of our nature simply in the sense that we have the propensity to judge in accordance with it (for an illuminating discussion of this view, see chapter one of Peregrin (2014)). In this sense, there could indeed be a rivalry of logics which could be decided to some degree empirically. Undoubtedly, the shape of our logic has to correspond in some way to how we use the *logical words* in our natural languages, that is words such as *therefore*, *not*, *or*, etc. Quinean holism also shows us that such a fundamental discipline as logic has to react to empirical findings. Indeed, some authors went very far down this path, notably Putnam who in Putnam (1968) argues in favour of quantum logic as a theory vindicated by empirical data. Yet still, holism also teaches us that the contact of logic with experience should not be very intense, the gains of choosing the path of empirization of geometry would hardly prevail over all the disadvantages and confusion engendered thereby. The holist picture actually does not serve to deny that disciplines such as logic are *a priori*, despite appearances. Understood in a correct way, it rather shows us exactly why they are so immune to revision, though an important point is also that no such immunity is absolute. Important as it is, this possibility of changing logic should not be overemphasized, as Quine himself might have had the tendency to do in *Two dogmas*.

What should restrain us from an excessive amount of iconoclasm and empiricism with regard to logic is that we ought to care for the intelligibility and purity of the very notion of logic. Why should we, after all, call something directly testable by linguistic empirical findings *logic*? Not that we are not free to do so, the question is rather whether we would thus not lose something deeper and more interesting. Indeed, succumbing to the empiricist suggestions means forgetting the anti-psychologistic lessons taught by, among others, Kant, Husserl and Frege. The last author surely was not just stubborn when he insisted that logic has nothing to do with psychology. Perhaps, taking in the holistic lesson, we should say that his strict division of logic and psychology is too radical, yet basically it points in a correct direction. Departing from the Quinean position that logic is something we just have the tendency to comply with, we should say that the necessity of logic is normative in the sense that we need it in order to have the status of a rational human being, capable of reasoning and argumentation. This is the sense in which we want to see logic as a part of human nature.

2.2 Appreciating the normativist lessons

Can we, then, say that one of the logics does what it should do and is thus closer to its true purpose than the other ones? Clearly, should logic for example lack the rule of *modus ponens*, we would probably be at loss as to why to regard it as logic at all (though even logics abandoning this law have been proposed). Yet, the situation is different with respect to other contentious laws. Can the law of the excluded middle fail to hold in some contexts? There seems to be no definitive answer. Perhaps it does not really belong to the very notion of logic to have or not to have such a law? Logic appears to be fulfilling a function which can be fulfilled in various ways, i.e. by various different logics, as, for example, by those asserting the validity of the law of the excluded middle and those not doing so. What could such a function be? In fact, can we say that logic has got any concrete purpose, at all?

It is indeed difficult to say what the purpose of logic should be, while it is not so difficult to adduce arguments to the conclusion that it is actually quite idle. Indeed, history knows of figures who doubted the meaningfulness of logic, René Descartes, in particular, is famous for this (see paragraph 6 of chapter 2 of Descartes (1965)).

Logic seems to be saying nothing but what we already knew. In a different context, Kant said that logic always comes too late (a very helpful exposition of Kant's surprisingly modern views on logic can be found in Wolff (1995)). Kosta Došen shows in a very illustrative and systematic manner how the rules of logic always articulate only that which was already present in the structure of our discourse (Došen (1989)). We cannot ingore these insights, yet there is a possibility to accept them while retaining the conviction that logic is still a useful discipline. Jaroslav Peregrin draws (in Peregrin (2014)) a distinction between what he calls tactical and constitutive rules of a given game. While the tactical rules tell us how to play the game smartly, so that we can win or be otherwise successful, the constitutive rules enable this game to come into existence. The common mistake when speaking about the purpose of logic and the rules it provides us about, e.g. the way the conditional works, is to regard these as tactical rules. This means that we believe that the rules of logic help us have more true beliefs, be more successful at argumentation, etc. Such views then result in disappointments of the kind we just exemplified.

According to Peregrin, then, logic does not deal with tactical rules but rather with the constitutive rules of our language games. These spell out what makes the various rule-governed practices we engage in into language games. But how can this be, given that we already know what logic tells us, as is shown in Došen's article? Can't we see that language games work very well, even without logic? The point is, though, that by enabling these language games to come into existence in the first place, the inferential moves sanctioned by logic typically cannot be a reasonable part of these games, they are too basic in comparison with the other inferential rules. Let us compare the rules of logic with other rules which have a similar status. Wittgenstein discusses (in Wittgenstein (1984)), among many other examples of the things that we should be particularly certain about, the fact that everyone is sure about the name one has. Indeed, this is one of the constitutive rules of many of the rule-governed practices (language games) we engage in. Yet, of course, typically reminding someone that it is very important to know one's own name would be comical. It would be a piece of advice given desperately late to be of any use. Similarly, should we try to advise somebody to augment the list of his beliefs by $A \to B$ whenever he knows that he can infer A from B, the reaction would be most likely to be that of puzzlement or amusement.

2.2.1 What is the purpose of logic, then?

The doctrine we are advocating here is that of logical expressivism that was introduced by Brandom in his *Making it explicit* (Brandom (1994)). Logic, according to this view, serves to make explicit the inferential rules that are implicit in our language. The usefulness of this explicitness is not hard to see, as we are thus enabled to reflect on these rules and perhaps also change them. Logic is, as Brandom puts it, an organ of our *semantic consciousness*. In order to make this approach viable we, together with Brandom, have to admit that an inference can be legitimate without being sanctioned by logic, which means without being in a form of a logically valid inference rule. An example of such an inference would be

$$\frac{\text{Rex is a dog}}{\text{Rex is a mammal}}$$

Indeed, this is a correct inference step, not an abbreviation of one, no enthymeme resulting from omitting a premise such as *All dogs are mammals*. Indeed, to formulate this sentence we would need to already use logical tools, such as the general quantifier and conditional[23]. This sentence states the rule we implicitly follow in our language as its competent users.

Denying the legitimacy of such inferential steps not sanctioned directly by logic invites the question as to which of the many logics we should consider as underlying and legitimizing all the inferences we do. The position that every correct instance of inference is such because it complies to logic (called by Brandom *formalism* in Brandom (2000)) makes us wonder which logic it should be. The fact that there are apparently no viable ways of arguing in favour of any of these logics seems to me to be a strong reason against this formalism and for actually embracing the Brandomian stance of acknowledging extra-logical inferential correctness. And when we accept the inferences of this kind as correct, it is then meaningful to speak of expressing rules which found their correctness by the means of logic (instead of their being validated by logic in the first place). We can thus say that logic is capable of making the rules of inference explicit.

But still, can we say that logic is being done with one certain purpose, can we sum up all the logics we have under the heading of any purpose at all? And if the answer is yes, how can we be sure that we have found the true purpose and thus are trying to decide which logic is more logical than others by correct criteria? Obviously, the logicians who create new logics do not first ask themselves whether the new logic can in fact fulfill some particular general purpose, such as making inferential rules explicit. Yet, Brandom identified how logic and in particular the logical expressions such as *if, then, therefore, it is not true that*, etc. work in natural language. He described something like a natural logic. Yet, it is hardly thinkable that any of the mathematical logics could behave just like the natural logic we use. In fact, speaking of one natural logic is an abstraction because every natural language together with its logical vocabulary is a flux and the rules of it are implicit and unstable. The rules of logic, on the other hand, are very precise and articulated.

[23] We would obviously formalize this sentence as $\forall x(Dog(x) \to Mammal(x))$.

Thus, there is a fundamental discrepancy with the original purpose of logic and the logical systems we know so that it makes no sense to say that one of them is some sense fulfilling the purpose of logic better than the others. The logical systems form a scientific, systematic prolongation of the natural logic which serves to express the inferential relationships which are the very core of our language games, or at least those fundamental ones which include giving and asking for reasons. As prolongations they are fundamentally dependent on this original activity of ours, they cannot begin anew in some pure or perfect manner.

As we know from Quine, "science is a continuation of common sense" (Quine (1951)). The language of science does not work in a fundamentally different way than the natural language, yet it puts special emphasis on exactness. Thus, it not only creates its specific terminology but also can make very good use of a stricter logic. Yet the study of how the logic of science and particularly of mathematics[24] could serve as the tool of expression of inferential rules in science led to the construction of different systems which can all plausibly fulfill the expressivist role. Here it is good to note that the term *implicit inferential rule* has to be taken with a grain of salt. One of the features of an implicit inferential rule is its high degree of indefiniteness. It often bears the potential to be converted into various different explicit rules which contradict each other. The explicit statement of the rule is thus an act of creation, as well. We do not just retell explicitly what was already present implicitly, we reshape it as well. This means that there are indeed more possibilities as to which rules to state and accept explicitly and still in fact see them as a continuation of the practices guided by the implicit rules that existed before.

2.3 Two freedoms of choice

Calling the approach to logic proposed by Brandom *logical expressivism* is basically right and suggestive but, we have to remember that we are not merely expressing something, which is simply already there, we are also thereby giving it a shape. There is this fundamental freedom to formulate various inferential rules on the basis of the workings of the already functioning discourse. To be sure, this is by no means an anarchy, the discursive practices determine what may be regarded as an expression of the rules which govern them to a great degree but not absolutely. I believe expression is a much bigger part of the enterprise than stipulation and therefore it is basically correct to speak of logical expressivism.

Nevertheless, the creative element present in formulating the rules of inference is very important and we have seen two kinds of the (very restricted) freedom we have when we formulate inference rules. Firstly, we can choose the exact shape of the rules and thus the logical relationships between propositions and thus between concepts. For example, consider the following potential law of inference in biology:

$$\frac{\text{XYZ is a dog}}{\text{XYZ has got lungs}}$$

[24] There is no reason to think that all the sciences share the same needs, not all of them are served by logic as much as mathematics.

It appears to be indeterminate whether we should regard this rule as valid or otherwise. Certainly the ordinary talk about dogs and possibly also the discourse of scientific zoology leaves this under-determined and we can thus choose. If we encountered animals which shared with dogs as we know them all their properties, except that they lacked lungs, we would hardly be sure whether we should consider them as a specific kind of dog or rather as a different category of animal. Of course, we do not have to do this if this particular decision is not of great importance to us. As unimportant as this individual decision may be, language as we know it is formed by our capacity to make lots of such decisions and our activity of formulating rules thus involves the kind of freedom we are talking about. The first freedom thus consists in our (limited) leeway in specifying which sentences are in logical relationships. Here, we just saw an example of two sentences which were indeterminate as to whether the first entailed the other one.

It is nevertheless the other kind of freedom that is important for us now, namely our freedom to choose the tools of this expression and formulation of inferential rules, i.e. our logic. In the nineteenth and twentieth century, mathematical methods were developed which enabled us to construct various logics. Every such logic has got many properties, some of which are postulated and some of which discovered by the mathematical logicians. Because of differences in these properties (such as compactness, completeness, decidability, etc.), the various logics can be seen as different kinds of an instrument of expressing the inferential rules, i.e. of *making them explicit*. Just as we can have many different hammers which can be useful for different purposes, so we can have various logics that express the inferential rules in our language games. It is true that by constructing new kinds of hammers we can also use them for quite novel purposes, thus partly changing our original conceptions of what a hammer is for. Thus, when logicians develop new logics, we can get new ideas about what logic can be useful for. Yet, just as at a certain point it is not meaningful to call something a hammer because it is perhaps too big, so it might be unreasonable to call certain mathematical theories *logics* because they cannot be reasonably seen to serve the purpose of expressing inferential relationships, even though they contain elements which behave similarly as, say, the connectives of classical logic. Nevertheless, the development of new logics as well as further mathematical discoveries about those we already know can bring us to a new understanding of what it means to make the rules guiding discursive practices explicit.

Summing up my position, I claim that there is a mutual influence between the philosophical reflection of logic and its purpose and the mathematical study and creation of new systems thereof. This mutual influence is a motor of development of logic as a discipline in general. Despite this dynamic relationship, there are still boundaries as to what can reasonably be regarded as logic, namely a system which can make explicit the inferential rules in our language games.

3. Back to geometry

Though our primary focus is on the plurality of logics, we also discussed the plurality of geometries, hoping to find useful lessons about logic therein. So far, we have arrived at logical expressivism as a philosophical stance which enables us to explain the plurality of logics thanks to its analysis of what a logic should actually do. We saw that logical expressivism is built upon the notion of inference, which is correct without being sanctioned by logic. In other words – material, as opposed to formal, inference. Acceptance of this notion can, on the other hand, also be made plausible by the plurality of logics. Logical pluralism and logical expressvism thus support each other. In the case of geometry we arrived primarily at holism and thereby saw that various geometries can be a part of broader theories helping us deal with the world. Limited holism is also a reasonable position for the philosophy of logic. The relationship between logic and mathematics can, particularly, be shaped in various ways, giving us leeway to choose whether to make adjustments rather in logic or in mathematics and in particular decide how much mathematics should be a part of logic. Yet, this holism should respect the expressivist role of logic in order to still be able to use the word *logic* meaningfully at all.

Now the holism we proposed for geometry was left rather abstract and general. This means that it does not provide much help for understanding what the specific task of this discipline is as opposed to, e.g. mechanics which interrelates closely with it. Consequently, we have so far arrived at no demarcation or at least guidelines for drawing a demarcation between what is and what is not a geometry. Nevertheless, Euclidian geometry truly deserves to be called geometry if anything does. The hyperbolic and elliptic geometries were plausibly shown to deserve the same, yet where is the boundary, how much can we change these theories and still talk about alternative geometries? As in the case of logic, we will not give any concrete list of possible shapes geometry can take, i.e. a list of acceptable theories. Yet, logical expressivism was found to be a fruitful formulation of what criterion we should use when deciding what is and what is not a logic, namely whether it could serve to make inference rules explicit.

As we tried to further address the problem of logical pluralism by revisiting that of geometrical pluralism, let us now try to apply what we learned about logic to geometry as well. Let us try to push the analogy as far as possible and speak of geometrical expressivism. What would such a doctrine involve? And does it make sense in the first place? Recall that there is a tradition of metaphorical talk of logical space, as can be seen, e.g. in Wittgenstein (2003). One of the features of space is that it is necessary for the entities which inhabit it. Indeed, they cannot exist but inside it. It constitutes relations between these entities, such as being above one another, being on the right of one another, etc[25]. Similarly, in the logical space there are specific relationships such as being a consequence of, being incompatible with, being a conjunction of two different propositions, etc. And indeed these relationships are essential for the propositions to be propositions at all. To disambiguate a little, when we have three physical objects, say a chair, a table and a ball, then the ball does not have to be between the other two, the chair does not have to be in front of the table, the ball on the table, etc. Yet clearly they have to be capable of entering into such relationships in order to be material objects all. In the same manner, when we speak of three propositions A, B and C, then A clearly does not have to be a consequence of B, C, its negation or perhaps the conjunction of both B and C. Yet, they have to be capable of entering such relations, though this time – as opposed to the physical objects – not between each other but perhaps with different propositions altogether.

[25] These spatial relations make sense, of course, only from a particular spectator's perspective.

Thus necessity and essentiality are among the important features physical and logical space share. The other is that they are in fact implicit, in a way invisible. We clearly see just the physical objects in the space, not the space itself. The logical relations between propositions and indeed their respective positions in the logical space is something we also do not realize primarily and they indeed have to be made explicit if we want to reason about them. Making both the logical and physical space explicit is not always useful. Indeed, it is called for when we encounter problems with the relations we are aware of just implicitly, when we cannot just go on in our practices, but have to analyze the situation more theoretically (we mentioned the somewhat trivial example with dogs and lungs but better ones could easily be provided). The logical analysis is typically useful (and sometimes even necessary) when we are at a loss as to whether a certain argument should be regarded as valid. We already saw that this also involves some freedom on our side. The need for making the spatial relationships explicit arises typically when we want to prevent some unfortunate accidents such as collapse of a bridge or when we react to those accidents which have already happened. Here we can apparently also speak of the twofold freedom we discovered when reflecting on similar features of logic. Given a logic, we saw that we have freedom how to articulate the rules by its means. On the other hand, there is also some freedom in choosing the logic we use for the formulation.

Similarly, there is some freedom in how we articulate the spatial relations between objects, ending possibly in (at least slightly) incompatible formulations. More importantly for us, though, there is a freedom to choose different geometries for the purposes of this articulation. Just as the ordinary practices of giving and asking for reasons do not force on us whether some logical laws, such as the excluded middle, hold, so we can also say that the ordinary practice of articulating the spatial relationships leaves under-determined at least whether the parallel postulate holds. We need to make a decision concerning this and similar issues only when the natural practice, the *natural geometry*, does not suffice, e.g. when human lives might be put at risk when bridges are built or when we are otherwise extremely dependent on the exactness of our understanding of geometrical notions (as in astronomy where we have to compensate for our inability to physically approach the examined objects).

4. Summary

We have seen that both the problem of the plurality of geometries and that of plurality of logics can be understood by seeing their role as that of expressing implicit spatial and logical relationships. In fact, we did not arrive at a specific demarcation either of logic or of geometry, but tried to understand what the nature of those disciplines is in the light of the plurality which cannot be denied. Considering the relation, or rather the gap, between the mathematical systems and the practices which underlie them, we see that the attempts at demarcation have indeed little hope of helping us understand geometry or logic much more deeply.

Acknowledgments

This article was supported by grant 17-15645S *Logical models of reasoning and argumentation in natural language*, led by Jaroslav Peregrin from Czech Academy of Sciences.

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